Challenge of the Week

September 9–September 22, 2008

Problem

Reconstruct the multiplication: (Letters A, B, C, D, E, and F stand for particular digits, asterisks may be any digit.)

```
            A B C D E F
          * * * * * * *
          -------------
            A B C D E F
          F A B C D E
        E F A B C D
      D E F A B C
    C D E F A B
  B C D E F A
A B C D E F
-----------------------
A * * B * C D * * E * F
```

Solution

Here is the reconstruction:

```
            1 4 2 8 5 7
          1 3 2 6 4 5 1
          -------------
            1 4 2 8 5 7
          7 1 4 2 8 5
        5 7 1 4 2 8
      8 5 7 1 4 2
    2 8 5 7 1 4
  4 2 8 5 7 1
1 4 2 8 5 7
-----------------------
1 8 9 4 9 2 8 1 0 5 0 7
```

The number 142857 has nice cyclic properties, coming from the decimal expansion for 1/7:

$$
\begin{array}{ll}
1/7 = 0.\overline{142857} & 1 \cdot 142857 = 142857 \\
3/7 = 0.\overline{428571} & 3 \cdot 142857 = 428571 \\
2/7 = 0.\overline{285714} & 2 \cdot 142857 = 285714 \\
6/7 = 0.\overline{857142} & 6 \cdot 142857 = 857142 \\
4/7 = 0.\overline{571428} & 4 \cdot 142857 = 571428 \\
5/7 = 0.\overline{714285} & 5 \cdot 142857 = 714285
\end{array}
$$

Many solvers were already familliar with the number 142857, and guessed ABCDEF immediately. From the table above, we can see that the second multiplicand must be 1326451. Finally, carrying out the multiplication gives the last line 189492810507.

If we didn't know about 142857, how could we find it? (Also, how do we ensure that there aren't other solutions?) The following analysis is due to Lloyd Sakazaki:

Consider the multiplication corresponding to the second row of the intermediate calculation: ABCDEF $\cdot\, m$ = FABCDE (where m is a single digit $2, 3, \ldots, 9$). Letting Z = ABCDE, we can write this multiplication as

$$\begin{aligned} \text{ABCDEF} \cdot m &= \text{FABCDE} \\ (10Z + \text{F}) \cdot m &= 100000\text{F} + Z \qquad \text{which simplifies to} \\ (10m - 1)Z &= (100000 - m)\text{F} \end{aligned}$$

Trying each possiblity for m, we get:

$m = 2$:	$19Z = 99998\text{F}$	no solution since 99998 is relatively prime to 19
$m = 3$:	$29Z = 99997\text{F}$	no solution since 99997 is relatively prime to 29
$m = 4$:	$39Z = 99996\text{F}$	simplifies to $Z = 2564\text{F}$; possible solutions
$m = 5$:	$49Z = 99995\text{F}$	simplifies to $7Z = 14285\text{F}$; possible solution $Z = 14285$, F = 7
$m = 6$:	$59Z = 99994\text{F}$	no solution since 99994 is relatively prime to 59
$m = 7$:	$69Z = 99993\text{F}$	simplifies to $23Z = 33331\text{F}$; no solution since 33331 is relatively prime to 23
$m = 8$:	$79Z = 99992\text{F}$	no solution since 99992 is relatively prime to 79
$m = 9$:	$89Z = 99991\text{F}$	no solution since 99991 is relatively prime to 89

Only cases $m = 4$ and $m = 5$ give possible solutions. First, we examine $m = 4$ more closely by looking at its subcases:

F	Z	ABCDEF
0	$2564 \cdot 0 = 0$	000000
1	$2564 \cdot 1 = 2564$	025641
2	$2564 \cdot 2 = 5128$	051282
3	$2564 \cdot 3 = 7692$	076923
4	$2564 \cdot 4 = 10256$	102564
5	$2564 \cdot 5 = 12820$	128205
6	$2564 \cdot 6 = 15384$	153846
7	$2564 \cdot 7 = 17948$	179487
8	$2564 \cdot 8 = 20512$	205128
9	$2564 \cdot 9 = 23076$	230769

For each of the subcases $\text{F} = 1, 2, \ldots, 9$, multiplication of ABCDEF by numbers in the set $\{2, 3, \ldots, 9\}$ does not produce the five cyclic permutations—FABCDE, EFABCD, DEFABC, CDEFAB and BCDEFA—that are required by the specified structure of the multiplication.

Only the case with $m = 5$, $Z = 14285$, $\text{F} = 7$ remains. This leads to the unique solution given above with ABCDEF = 142857.