

Challenge Of the Week

September 23—September 29, 2008

Problem

The sequence $S = 1, 0, 1, 0, 1, 0, 3, 5, 0, 9, \dots$ is generated by starting with “1, 0, 1, 0, 1, 0”; every successive number is found by adding the previous six numbers and taking the last digit. Prove that the pattern “0, 1, 0, 1, 0, 1” will never appear in S .

Solution 1

This is the most straightforward solution, though tedious.

If the numbers in the sequence are, $S = s_1, s_2, s_3, \dots$, then the rule for generating the number s_{i+6} given the previous six numbers is then to choose the (unique) number between 0 and 9 so that $s_{i+6} = s_i + s_{i+1} + s_{i+2} + s_{i+3} + s_{i+4} + s_{i+5} \pmod{10}$.

If the pattern “0, 1, 0, 1, 0, 1” ever appears, then in fact it must appear if we worked modulo 5. This works because if $s_i = 0$ or $1 \pmod{10}$, then $s_i = 0$ or $1 \pmod{5}$. Contrapositively, if “0, 1, 0, 1, 0, 1” doesn’t appear when computing the sequence modulo 5, it cannot appear in the original sequence. We show that it can’t appear modulo 5.

Just to start computing the sequence and see what happens! (This is best done by computer, but it’s actually not hard to do by hand.)

$S = 1, 0, 1, 0, 1, 0, 3, 0, 0, 4, 3, 0, 0, 2, 4, 3, 2, 1, 2, 4, 1, 3, 3, 4, 2, 2, 0, 4, 0,$
 $2, 0, 3, 4, 3, 2, 4, 1, 2, 1, 3, 3, 4, 4, 2, 2, 3, 3, 3, 2, 0, 3, 4, 0, 2, 1, 0, 0, 2,$
 $0, 0, 3, 0, 0, 0, 3, 1, 2, 1, 2, 4, 3, 3, 0, 3, 0, 3, 2, 1, 4, 3, 3, 1, 4, 1, 1, 3, 3,$
 $3, 0, 1, 1, 1, 4, 0, 2, 4, 2, 3, 0, 1, 2, 2, 0, 3, 3, 1, 1, 0, 3, 1, 4, 0, 4, 2, 4, 0,$
 $4, 4, 3, 2, 2, 0, 0, 1, 3, 3, 4, 1, 2, 4, 2, 1, 4, 4, 2, 2, 0, 3, 0, 1, 3, 4, 1, 2, 1,$
 $2, 3, 3, 2, 3, 4, 2, 2, 1, 4, 1, 4, 4, 1, 0, 4, 4, 2, 0, 1, 1, 2, 0, 1, 0, 0, 4, 2, 2,$
 $4, 2, 4, 3, 2, 2, 0, 3, 2, 1, 0, 3, 4, 3, 3, 4, 2, 4, 0, 1, 4, 0, 1, 0, 1, 2, 3, 2,$
 $4, 2, 4, 2, 2, 1, 0, 1, 0, 1, 0, 3, 0, \dots$

Observe that the sequence (created modulo 5) repeats itself without ever containing the pattern “0, 1, 0, 1, 0, 1”. So the pattern can never appear in the sequence modulo 10, either.

(Note that we could have just computed terms from the original sequence modulo 10 until it repeated, but this takes much longer.)

Solution 2

We show that “0, 1, 0, 1, 0, 1” will never appear by finding an invariant property of the numbers as we step along the sequence. Again label the numbers in the sequence $S = s_1, s_2, s_3, \dots$. Now define¹ the function

$$Q(i) = 2s_i + 4s_{i+1} + 6s_{i+2} + 8s_{i+3} + 2s_{i+5} \pmod{10}$$

For example, at the beginning of the sequence, we get $Q(1) = 2 \cdot 1 + 4 \cdot 0 + 6 \cdot 1 + 8 \cdot 0 + 2 \cdot 0 = 8 \pmod{10}$.

I claim that $Q(i+1) = Q(i)$. To check this, just compute:

$$\begin{aligned} Q(i+1) &= 2s_{i+1} + 4s_{i+2} + 6s_{i+3} + 8s_{i+4} + 2s_{i+6} \pmod{10} \\ &= 2s_{i+1} + 4s_{i+2} + 6s_{i+3} + 8s_{i+4} + 2(s_i + s_{i+1} + s_{i+2} + s_{i+3} + s_{i+4} + s_{i+5}) \pmod{10} \\ &= 2s_i + 4s_{i+1} + 6s_{i+2} + 8s_{i+3} + 10s_{i+4} + 2s_{i+5} \pmod{10} \\ &= 2s_i + 4s_{i+1} + 6s_{i+2} + 8s_{i+3} + 2s_{i+5} \pmod{10} \\ &= Q(i) \end{aligned}$$

Since the beginning of the sequence gave $Q(1) = 8$, this means that $Q(i) = 8$ for all i .

Now suppose that the sequence contained the terms “0, 1, 0, 1, 0, 1”; that is, there is some k so that $(s_k, s_{k+1}, s_{k+2}, s_{k+3}, s_{k+4}, s_{k+5}) = (0, 1, 0, 1, 0, 1)$. Then we compute $Q(k) = 2 \cdot 0 + 4 \cdot 1 + 6 \cdot 0 + 8 \cdot 1 + 2 \cdot 1 \pmod{10} = 4$. But this is impossible, since Q is always 8.

Solution 3

(Adaped from Payush Jain) This solution will appeal to those who know some linear algebra.

The rule for generating numbers in the sequence is nicely encoded as a matrix equation:

$$\begin{bmatrix} s_{i+1} \\ s_{i+2} \\ s_{i+3} \\ s_{i+4} \\ s_{i+5} \\ s_{i+6} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_i \\ s_{i+1} \\ s_{i+2} \\ s_{i+3} \\ s_{i+4} \\ s_{i+5} \end{bmatrix} \pmod{10}$$

¹How could you guess such a function? One way is to guess the general form of it, and solve for the coefficients so that the claim works out.

Let's call the 0-1 matrix above A . From here, it's not too big a leap to define the function

$$M(i) = \begin{bmatrix} s_i & s_{i+1} & s_{i+2} & s_{i+3} & s_{i+4} & s_{i+5} \\ s_{i+1} & s_{i+2} & s_{i+3} & s_{i+4} & s_{i+5} & s_{i+6} \\ s_{i+2} & s_{i+3} & s_{i+4} & s_{i+5} & s_{i+6} & s_{i+7} \\ s_{i+3} & s_{i+4} & s_{i+5} & s_{i+6} & s_{i+7} & s_{i+8} \\ s_{i+4} & s_{i+5} & s_{i+6} & s_{i+7} & s_{i+8} & s_{i+9} \\ s_{i+5} & s_{i+6} & s_{i+7} & s_{i+8} & s_{i+9} & s_{i+10} \end{bmatrix}$$

which satisfies the matrix equation $M(i+1) = AM(i)$, i.e., $M(i) = A^{i-1}M(1)$.

Taking determinants, we get that $\det M(i) = \det(A^{i-1}M(1)) = \det(A)^{i-1} \det(M(1))$. We compute $\det A = -1$ and

$$\det(M_1) = \det \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 & 4 & 3 \end{bmatrix} = 4 \pmod{10}.$$

We conclude that $\det M(i) = \pm 4 \pmod{10}$ for all i .

Now suppose that the sequence contained the terms “0, 1, 0, 1, 0, 1”; that is, there is some k so that $(s_k, s_{k+1}, s_{k+2}, s_{k+3}, s_{k+4}, s_{k+5}) = (0, 1, 0, 1, 0, 1)$. We compute the next few terms of the sequence to be 3, 6, 1, 2, 3. Then we compute

$$\det M(k) = \det \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 3 & 6 \\ 1 & 0 & 1 & 3 & 6 & 1 \\ 0 & 1 & 3 & 6 & 1 & 2 \\ 1 & 3 & 6 & 1 & 2 & 3 \end{bmatrix} = 3 \pmod{10}$$

But this is impossible, since $\det M(i)$ is always $\pm 4 \pmod{10}$.