Challenge of the Week

September 30–October 6, 2008

Problem

Find positive integers $x_1, x_2, \ldots, x_m$ so that

1. the sum $x_1 + x_2 + \ldots + x_m = 2008$.
2. the product $x_1 x_2 \cdots x_m$ is as large as possible.

What is the maximum possible value for their product? (For example, the maximum product is at least as large as 2006 since we can pick three numbers: $x_1 = x_2 = 1$ and $x_3 = 2006$.)

Solution

(Contributed by Jack Lee, slightly edited here.)

Answer: The maximum value of the product is $4 \cdot 3^{668}$, realized by the sequence $(2, 2, 3, \ldots, 3)$ with two 2's and 668 3's.

Proof: Since there are only finitely many sequences of positive integers that sum to 2008, there must be a sequence that realizes the maximum product. Let $(x_1, \ldots, x_m)$ be such a sequence.

There are no 1's in the sequence. If there's a 1, take any other number x_j in the sequence and replace the pair $(1, x_j)$ by $(x_j + 1)$. This gives the same sum but a larger product.

The sequence contains no entries greater than 4. If x_i were greater than 4, then we could replace it by the pair $(x_i - 2, 2)$ without changing the sum. This replaces x_i in the product by $2(x_i - 2) > x_i$ to make a larger product.

We may assume that the sequence contains no 4's. If there is a 4, we can replace it by a pair of 2's without changing either the product or the sum.

There are no more than two 2's in the sequence. If there were at least three 2's, we could replace $(2, 2, 2)$ by $(3, 3)$ to get the same sum but a larger product.

Combining the above results, we conclude that the sequence consists of at most two 2's and the rest 3's. Because $2008 \equiv 1 \pmod 3$, the only possibility is that there are exactly two 2's and 668 3's, so the product is $2^2 \cdot 3^{668}$.