

Challenge of the Week

October 14–October 20, 2008

Problem

Suppose there's a row of n lights, which (simultaneously) turn on and off each minute according to the following rules:

- If a light is currently on, then the next minute it will turn off.
- If a light is currently off, and exactly one light adjacent to it is currently on, then the next minute it will turn on. Otherwise it will stay off.

For some n , there are on/off configurations so that there will always be at least one light on. For example, when $n = 2$, start with [on,off]; then the lights will follow the pattern [on,off] \rightarrow [off,on] \rightarrow [on,off] $\rightarrow \dots$. There are some n where no matter how we start the pattern of lights, eventually all the lights turn off permanently.

The problem is to find all possible (or as many as possible) values of n , and corresponding starting configurations, so that there will always be some light on.

Solution

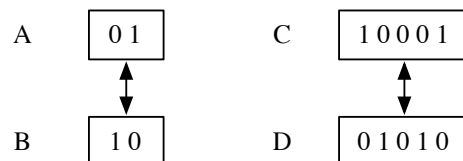
Starting configurations exist for all n except $n = 1$ and $n = 3$.

For brevity, denote an “on” light with a 1, and an “off” light with a 0.

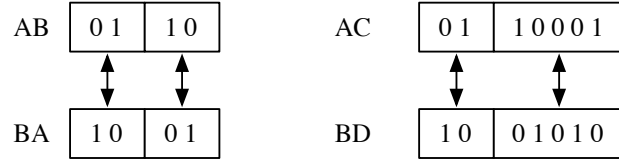
Claim 1: If $n \neq 1, 3$, there exists a configuration.

Proof: The key idea here is to find patterns that repeat quickly for small n , and then glue such patterns together for larger n .

There exist configurations for $n = 2$ and $n = 5$. We show these configurations below:



Now, we can glue these four together to make longer patterns. In particular, starting with A and B , we can make the patterns AB and BA . Also, we can make the patterns AC and BD . To verify this, just check:



Finally, by repeatedly gluing $ABABAB..$ we can get configurations for all even numbers. Given such a configuration, we can then glue either C or D at the end (depending on whether the configuration ends with A or B) to get patterns for all odd numbers greater than or equal to 5.

Claim 2: When $n = 1$ or 3 , there are no configurations that work.

Proof: When $n = 1$, it is trivial to check that after at most one turn, the single light will be permanently off. When $n = 3$, there are few enough cases to check them all; the diagram below shows how every configuration when $n = 3$ reduces to the all off configuration.

