Challenge of the Week

October 28–November 3, 2008

Problem:

Prove that for any positive integer n the following inequality holds:

$$\sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{n}}}} < 3$$

Solution 1:

(Due to Alberto Chiecchio)

Define the function $\phi(n) = \sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{n}}}} = 2^{1/2}3^{1/4}4^{1/8}\cdots n^{1/2^{n-1}}$.

Observe that we can write ϕ recursively:

$$\phi(2) = \sqrt{2}$$
$$\phi(n) = \phi(n-1)n^{1/2^{n-1}}.$$

Claim: $\phi(n) < \dfrac{3}{(n+2)^{1/2^{n-1}}}$.

Proof: We argue by induction. When $n = 2$, we have $\phi(2) = \sqrt{2} < \frac{3}{4^{1/2}}$. Now, suppose the claim is true when $n = k-1$; we wish to show the claim holds for $n = k$. We compute:

$$\begin{aligned}
\phi(k) &= \phi(k-1)k^{1/2^{k-1}} \\
&< \frac{3}{(k+1)^{1/2^{k-2}}}k^{1/2^{k-1}} \\
&= 3\left(\frac{k^{1/2}}{k+1}\right)^{1/2^{k-2}} \\
&< 3\left(\frac{1}{(k+2)^{1/2}}\right)^{1/2^{k-2}} \qquad (*) \\
&= \frac{3}{(k+2)^{1/2^{k-1}}}.
\end{aligned}$$

The step $(*)$ holds because $k(k+2) = k^2+2k < k^2+2k+1 = (k+1)^2$, or equivalently, $\frac{\sqrt{k}}{k+1} < \frac{1}{\sqrt{k+2}}$. $\square$

Since for all positive n we have $(n+2)^{1/2^{n-1}} > 1^{1/2^{n-1}} = 1$, it follows that $\phi(n) < \dfrac{3}{(n+2)^{1/2^{n-1}}} < 3$.

Solution 2

We can rewrite the left hand side of the inequality as

$$\sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{n}}}} = 2^{1/2}3^{1/4}4^{1/8}\cdots n^{1/2^{n-1}}$$

So if we take the logs, we must show

$$L = 2^{-1}\log 2 + 2^{-2}\log 3 + 2^{-3}\log 4 + \cdots + 2^{1-n}\log n < \log 3.$$

Now we can estimate the left hand side L as follows:

$$\begin{aligned}
L &= \sum_{k=1}^{n-1} \frac{1}{2^k}\log(k+1) \\
&\leq \sum_{k=1}^{\infty} \frac{1}{2^k}\log(k+1) \\
&= \sum_{k=1}^{7} \frac{1}{2^k}\log(k+1) + \sum_{k=8}^{\infty} \frac{1}{2^k}\log(k+1) \\
&\qquad \text{since } \log(k+1) \leq k \text{ for all } k, \text{ we can estimate the second sum as} \\
&\leq \sum_{k=1}^{7} \frac{1}{2^k}\log(k+1) + \sum_{k=8}^{\infty} \frac{1}{2^k}k\ dx
\end{aligned}$$

The sum on the right can be over-estimated with the integral

$$\int_8^\infty \frac{1}{2^{x-1}}(x-1)\ dx = \left.\frac{2^{1-x}(-\log(2)x + \log(2) - 1)}{(\log 2)^2}\right|_8^\infty = \frac{1+\log(128)}{128(\log 2)^2}$$

So that

$$L < \frac{\log 2}{2} + \frac{\log 3}{4} + \frac{\log 4}{8} + \frac{\log 5}{16} + \frac{\log 6}{32} + \frac{\log 7}{64} + \frac{\log 8}{128} + \frac{1+\log(128)}{128(\log 2)^2} = 1.092904 < 1.098612 = \log 3.$$