Challenge of the Week

November 11–November 17, 2008

Problem

Let x, y, and z be three positive real numbers such that $x+y+z=1$. Prove that $x^y y^z z^x \le \frac{1}{3}$.

Solution 1

Jacob Lewis found a very short solution using two high-powered inequalities. These can be found on Wikipedia.

Jensen's inequality is a general inequality for convex functions. Suppose ϕ is a convex function, and $x_1,\ldots,x_n$ and $\alpha_1,\ldots,\alpha_n$ are positive numbers. Define $\alpha = \sum_{i=1}^n \alpha_i$. Then

$$\sum_{i=1}^n \frac{\alpha_i}{\alpha}\phi(x_i) \le \phi\left(\frac{1}{\alpha}\sum_{i=1}^n \alpha_i x_i\right)$$

Using the function $\phi(x) = -\log(x)$, $n=3$, $x_1 = x, x_2 = y, x_3 = z, \alpha_1 = y, \alpha_2 = z, \alpha_3 = x$, and $\alpha = 1$, we get:

$$\log(x^y y^z z^x) = y\log(x) + z\log(y) + x\log(z) \le \log(xy+yz+xz)$$

Exponentiating this gives

$$x^y y^z z^x \le xy+yz+xz \tag{1}$$

Maclaurin's inequality. Let $S_k(x_1,\ldots,x_n)$ be the k-th elementary symmetric polynomial on n variables divided by $\binom{n}{k}$. Then for $1 \le i \le n-1$, we have

$$S_i^{1/i} \ge S_{i+1}^{1/(i+1)}.$$

In the particular case where $n=3, i=1$, this inequality states:

$$\left(\frac{x+y+z}{\binom{3}{1}}\right)^{1/1} \ge \left(\frac{xy+yz+xz}{\binom{3}{2}}\right)^{1/2} \qquad \text{or more simply}$$

$$\frac{x+y+z}{3} \ge \sqrt{\frac{xy+yz+xz}{3}}.$$

Since $x+y+z=1$, this means that

$$xy+zy+xz \le 1/3. \tag{2}$$

Combining (1) and (2) gives the result!

Solution 2

Here's an approach from the optimization point of view.

Define $f(x, y, z) = x^y y^z z^x$. We want to show that f gets no larger than 1/3 when $0 \leq x, y, z$ and $x + y + z = 1$.

Observe that when $x = 0$, $y = 0$, or $z = 0$, we have $f(x, y, z) = 0$. Thus we may assume that x, y, z are all strictly positive, so that $f(x, y, z)$ is also positive. We can now simplify the problem by taking logs. Let $g(x, y, z) = \log f(x, y, z) = y \log x + x \log y + x \log z$. Observe that f has a maximum whenever g has a maximum.

We can now use Lagrange multipliers to find where $g(x, y, z)$ is maximized on the constraint $x + y + z - 1 = 0$. The technique here is to define the Lagrangian

$$L(x, y, z, \lambda) = g(x, y, z) + \lambda(x + y + z - 1).$$

Any maxima or minima occur when the gradient of L is 0. We get

$$\begin{aligned}
\partial L/\partial x &= \frac{y}{x} + \log z + \lambda = 0 \\
\partial L/\partial y &= \frac{z}{y} + \log x + \lambda = 0 \\
\partial L/\partial z &= \frac{x}{z} + \log y + \lambda = 0 \\
\partial L/\partial \lambda &= x + y + z - 1 = 0
\end{aligned}$$

I claim the only solution to this equation is $x = y = z = \frac{1}{3}$. To show this, we may assume by the symmetry of the problem that there is a solution where x, y and z are not equal, then there is a solution where $x \leq y \leq z$. Subtracting the third equation from the first then gives

$$\begin{aligned}
0 &= \frac{y}{x} + \log z - \frac{x}{z} - \log y \\
&= \frac{yz - x^2}{xz} + \log\left(\frac{z}{y}\right)
\end{aligned}$$

The first term can only be 0 when $yz = x^2$, otherwise it will be positive. Similarly, the log term can only be 0 when $z = y$, otherwise it will be positive. We must conclude that $x = y = z$, and the last equation gives $x, y, z = 1/3$. Thus there is only one critical point to check.

When $x = y = z = 1/3$, we have $f(x, y, z) = 1/3$.

We know that this in fact a local maximum, since the function g is concave in x, y, and z and the constraint $x + y + z - 1 = 0$ is affine.