Challenge Of the Week

November 25—December 1, 2008

Problem

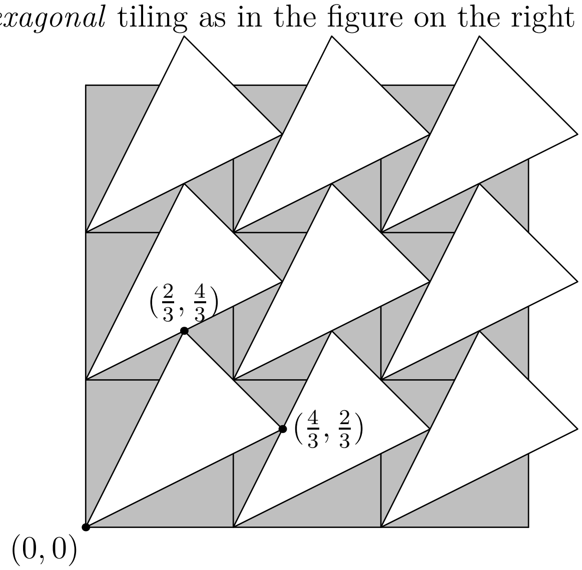
The vertices of a triangle T have coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3). Suppose that for any integers h and k, not both zero, the shifted triangle with vertices $(x_1 + h, y_1 + k)$, $(x_2 + h, y_2 + k)$, and $(x_3 + h, y_3 + k)$ has no common interior points with the original triangle. (That is, if you tiled the plane with shifted copies of triangle T, none of the interiors of the triangles would overlap.)

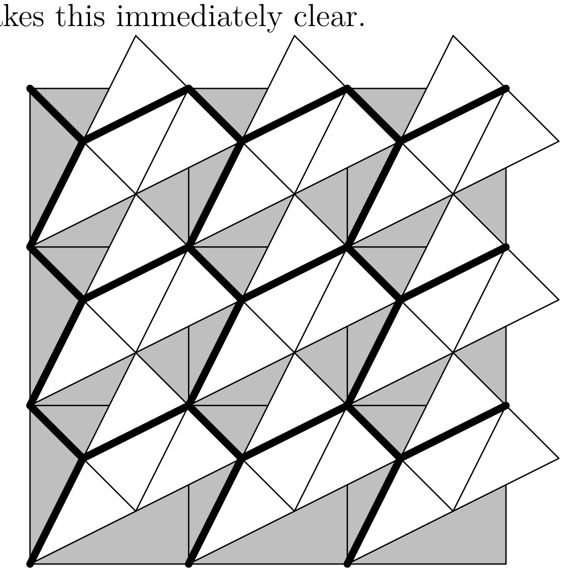
1. Is it possible for the area of triangle T to be greater than $1/2$?
2. What is the maximum possible area of triangle T?

Solution

The maximum possible area triangle T can have is $2/3$.

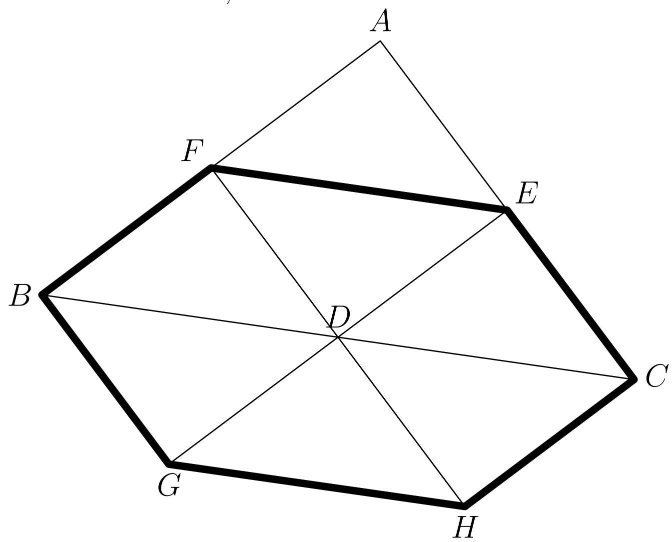
A triangle T with coordinates $(0, 0)$, $(\frac{4}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{4}{3})$ gives an example where the area is $2/3$. This can be seen in the figure on the left. It's not hard to verify that the triangle really does have area $2/3$ via direct calculation, but this is not necessary; viewing the tiling as a *hexagonal* tiling as in the figure on the right makes this immediately clear.





This hexagonal tiling also provides the motivation for a proof that it is impossible to do better.

Suppose we have any triangle ABC so that integral translates do not overlap. Define D, E, F to be the midpoints of BC, CA, and AB respectively, and extend ED to G and FD to H so that $ED = DG$ and $FD = DH$, as shown below.



We will show that integral translates of $BFECHG$ cannot have any common interior points. Thus the hexagon $BFECHG$ has area at most 1, and hence triangle ABC has area at most 2/3.

Lemma: Integral translates of $BFECHG$ have no common interior points.

Proof: Suppose to the contrary that $BFECHG$ has a common point with an integral translate $B'F'E'C'H'G'$. If an interior point overlaps, so must at least one of the vertices; since $A'B'C'$ has no common points with ABC, we know that the trapezoids $FEBC$ and $F'E'B'C'$ have no point in common. Thus we may assume in either E' or F' is inside the trapezoid $BGHC$.

If E' lies in triangle DBG, then B will be inside the integral translate $A'B'C'$ of ABC. Similarly, if F' is inside triangle DCH, then C will be inside $A'B'C'$. Finally, if either E' or F' is inside triangle DGH, then A' will be inside ABC. In any case, we arrive at a contradiction, so that the hexagons $BFECHG$ and $B'F'E'C'H'G'$ cannot overlap.