Challenge Of the Week

November 25—December 1, 2008

Problem

The vertices of a triangle $T$ have coordinates $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$. Suppose that for any integers $h$ and $k$, not both zero, the shifted triangle with vertices $(x_1 + h, y_1 + k)$, $(x_2 + h, y_2 + k)$, and $(x_3 + h, y_3 + k)$ has no common interior points with the original triangle. (That is, if you tiled the plane with shifted copies of triangle $T$, none of the interiors of the triangles would overlap.)

1. Is it possible for the area of triangle $T$ to be greater than 1/2?
2. What is the maximum possible area of triangle $T$?

Solution

The maximum possible area triangle $T$ can have is 2/3.

A triangle $T$ with coordinates $(0, 0)$, $(4/3, 2/3)$ and $(2/3, 4/3)$ gives an example where the area is 2/3. This can be seen in the figure on the left. It’s not hard to verify that the triangle really does have area 2/3 via direct calculation, but this is not necessary; viewing the tiling as a hexagonal tiling as in the figure on the right makes this immediately clear.

This hexagonal tiling also provides the motivation for a proof that it is impossible to do better.
Suppose we have any triangle $ABC$ so that integral translates do not overlap. Define $D$, $E$, $F$ to be the midpoints of $BC$, $CA$, and $AB$ respectively, and extend $ED$ to $G$ and $FD$ to $H$ so that $ED = DG$ and $FD = DH$, as shown below.

![Diagram of triangle ABC with midpoints D, E, F, and extended lines to G and H]

We will show that integral translates of $BFECHG$ cannot have any common interior points. Thus the hexagon $BFECHG$ has area at most 1, and hence triangle $ABC$ has area at most $2/3$.

**Lemma:** Integral translates of $BFECHG$ have no common interior points.

**Proof:** Suppose to the contrary that $BFECHG$ has a common point with an integral translate $B'F'E'C'H'G'$. If an interior point overlaps, so must at least one of the vertices; since $A'B'C'$ has no common points with $ABC$, we know that the trapezoids $FEB'C$ and $F'E'B'C'$ have no point in common. Thus we may assume in either $E'$ or $F'$ is inside the trapezoid $BGHC$.

If $E'$ lies in triangle $DBG$, then $B$ will be inside the integral translate $A'B'C'$ of $ABC$. Similarly, if $F'$ is inside triangle $DCH$, then $C$ will be inside $A'B'C'$. Finally, if either $E'$ or $F'$ is inside triangle $DGH$, then $A'$ will be inside $ABC$. In any case, we arrive at a contradiction, so that the hexagons $BFECHG$ and $B'F'E'C'H'G'$ cannot overlap.