

Challenge of the Week

December 2–December 8, 2008

Problem

What is the smallest perfect square that ends in the four digits 9009 (in base 10, as usual)? Can you find the solution without a brute-force computer search?

Solution

We're looking for a number n so that n^2 ends in 9009. The easiest way to do this is to build up digit-by-digit.

Write $n = \sum_{i=0}^k 10^i a_i$; that is, n has the decimal representation $n = a_k a_{k-1} \dots a_3 a_2 a_1 a_0$.

When we square n , the last digit is $a_0^2 \pmod{10}$, so that the last digit a_0 must be either 3 or 7.

Now look at the last two digits of n . We get

$$n^2 \equiv (10a_1 + a_0)^2 \equiv 20a_1a_0 + a_0^2 \equiv 09 \pmod{100}.$$

If $a_0 = 3$, then this reduces to $60a_1 \equiv 0 \pmod{100}$, so that $a_1 = 0$ or 5. Alternately, if $a_0 = 7$, this reduces to $40 + 40a_1 \equiv 0 \pmod{100}$, so that $a_1 = 4$ or 9. Now we know that n ends in one of $\{03, 53, 47, 97\}$.

Next look at the last three digits of n . Computing $n^2 \equiv 009 \pmod{1000}$, putting in each of the possibilities for a_1 and a_0 , (similar to above) we can find a_2 ; we get 8 possibilities for the last three digits: $\{003, 503, 253, 753, 247, 747, 497, 997\}$.

Finally, look at the last four digits of n . Computing $n^2 \equiv 9009 \pmod{10000}$, we can find a_3 . The possibilities that for the last four digits are $\{1503, 6503, 2753, 7753, 2247, 7247, 3497, 8497\}$.

Looking at all the solutions, we find $n = 1503$ is the smallest number such that n^2 ends in 9009.

Matt Inouye pointed out that while there are a fair number of cases to look at, they may be efficiently organized as follows:

