Challenge Of the Week

January 20—January 26, 2008

Problem:

Let $g(n)$ be the number of solutions (x, y, z) of $x+2y+3z = n$ with x, y, and z all nonnegative integers. Show that

$$\begin{aligned} g(0) &= 1 \\ g(1) &= 1 \\ g(2) &= 2 \\ g(n) &= g(n-3) + \left\lfloor \frac{n}{2} \right\rfloor + 1, \qquad n = 3, 4, 5, \ldots \end{aligned}$$

where

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{when } n \text{ is even} \\ \frac{n-1}{2} & \text{when } n \text{ is odd.} \end{cases}$$

Solution:

First, let's count the number of solutions to the simpler problem $x+2y = n$. For any integer y, $0 \le y \le \frac{n}{2}$, we can let $x = n - 2y$ to get a solution in nonnegative integers. As there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ ways to pick y, the number of solutions is also $\left\lfloor \frac{n}{2} \right\rfloor + 1$.

Now for the main problem. We find that

$x + 2y + 3z = 0$ has one solution, $(0, 0, 0)$;
$x + 2y + 3z = 1$ has one solution, $(1, 0, 0)$; and
$x + 2y + 3z = 2$ has two solutions, $(2, 0, 0)$ and $(0, 1, 0)$.

Thus $g(0) = g(1) = 1$ and $g(2) = 2$.

Suppose $n \ge 3$. Every solution (x, y, z) of $x + 2y + 3z = n$ with $z \ge 1$ corresponds to a solution $(x, y, z-1)$ of $x+2y+3z = n-3$; there are $g(n-3)$ of these. Otherwise, if $z = 0$, the problem reduces to the simpler case $x + 2y = n$ where there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ solutions. Thus the total number of solutions is

$$g(n) = g(n-3) + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$