Challenge Of the Week

February 3—February 9, 2008

Problem:

At a high school, the lockers are in a long rectangular array, with 3 rows of N lockers each, where N is between 400 and 450. Originally, the lockers on the top row were numbered 1 to N, the middle row numbered from $N+1$ to $2N$, and the bottom row $2N+1$ to $3N$, all from left to right.

However, one evening the administration changed around the locker numbers so that the first column on the left is now numbered 1 to 3, the second column 4 to 6, and so forth, all from top to bottom.

Three friends, whose lockers are located one in each row, came in the next morning to discover that each of them now has the locker number that used to belong to one of the others.

What are their locker numbers, assuming that they all are three-digit numbers?

Solution:

The locker numbers are 246, 736, and 932.

To show this, first consider a single locker. We can write the original locker number as $iN+j$, where $0 \leq i \leq 2$ is the row and $1 \leq j \leq N$ is the column. After the change, this locker will be numbered $3(j-1)+(i+1) = 3j+i-2$.

Now consider the three friends' lockers. Since the lockers are located one in each row, we can write their numbers (before the change) as j_1, $N+j_2$, and $2N+j_3$, where $1 \leq j_1, j_2, j_3 \leq N$. For each of these lockers, the corresponding new locker numbers will be $3j_1-2$, $3j_2-1$ and $3j_3$.

There are two possibilities for how their original and new locker numbers were interchanged. If the friends originally had locker numbers A, B, C, then they can have new locker numbers C, A, B or B, C, A.

Case 1. The first possibility is that the numbers change as

$$j_1 = 3j_3, \qquad N+j_2 = 3j_1-2, \qquad 2N+j_3 = 3j_2-1.$$

Substituting the first equation into the second and solving for j_2, we get $j_2 = 9j_3 - 2 - N$. Substituting this into the last equality and solving for j_3, we get

$$j_3 = \frac{5N + 7}{26}$$

for this division to make sense, we must have $N \equiv 9 \pmod{26}$. The only possible value for N between 400 and 450 is $N = 425$. From here, we can solve for j_3, j_2, and j_1 to get $j_1 = 246$, $j_2 = 311$, and $j_3 = 82$. This means that the original locker numbers are 246, 736, and 932, which are changed to 736, 932, and 246.

Case 2. The second possibility is that the numbers change as

$$j_1 = 3j_2 - 1, \qquad N + j_2 = 3j_3, \qquad 2N + j_3 = 3j_1 - 2.$$

Similar computation as above gives

$$j_3 = \frac{11N + 5}{26}$$

for j_3 to be an integer, we must (again!) have $N \equiv 9 \pmod{26}$, so that $N = 425$. Solving for the $j's$, we get $j_1 = 344, j_2 = 115,$, and $j_3 = 180$. In this case the lockers will be numbered 344, 540 and 1030; the largest is too big for our assumptions, so this case does not happen.