Challenge of the Week

February 10–February 16, 2009

Problem

25 people go to a party on a cold winter day and leave their coats at the door. The first person to leave is in a rush and grabs one of the 25 coats at random. From there, as people leave, they either take their own coat if it is there, or grab a coat at random if their coat is not there.

What are the chances that the last person will take his own coat?

Solution

The probability that the last person will take his own coat is $1/2$. Perhaps surprisingly, this does not depend on the number (≥ 2) of people.

Let $P(n)$ be the chance that the last person of n will get his own coat when the first person chooses randomly. We show by induction that $P(n) = 1/2$ for $n \geq 2$.

Suppose first that there are only two people. Here, it is clear that the second person has probability $1/2$ of getting his coat, so $P(2) = 1/2$.

Suppose next that $P(2) = \ldots = P(n-1) = 1/2$; we will show that $P(n) = 1/2$.

Label the people $1, 2, \ldots, n$ by the order they leave. (So person 1 is the person who grabs a random coat and person n is the final person to leave.) There are three cases for what the first person may do:

1. He grabs his own coat. Then everyone else, including the last person, gets their own coat as well. This event happens with probability $1/n$.
2. He grabs the last person's coat, so the last person cannot get his coat. This event happens with probability $1/n$.
3. He grabs the jth person's coat, (for $j = 2, \ldots, n-1$). (Each possibility happens with probability $1/n$.) Then the people $2, 3, \ldots, j-1$ get their own coats, and person j is forced to pick a coat randomly, from the set of coats $\{1, j+1, j+2, \ldots, n\}$. Pretending that the first person's coat belongs to person j we are now in the same situation as the original problem, but with fewer people. By induction, the probability the last person will get his coat is $1/2$.

Summing over the different cases, we find

$$\begin{aligned} P(n) &= \underbrace{1 \cdot \frac{1}{n}}_{\text{first}} + \underbrace{0 \cdot \frac{1}{n}}_{\text{last}} + \sum_{j=2}^{n-1} \underbrace{\frac{1}{2} \cdot \frac{1}{n}}_{j\text{th}} \\ &= \frac{1}{n}\left(1 + \sum_{j=2}^{n-1} \frac{1}{2}\right) \\ &= \frac{1}{n}\left(1 + (n-2)\frac{1}{2}\right) \\ &= \frac{1}{2}. \end{aligned}$$

which completes the argument.