Challenge Of the Week

February 17—February 23, 2008

Problem:

Let $f(x) = x^4 + ax^3 + bx^2 + cx - c$. Suppose that f shares two distinct integral roots with its derivative $f'(x)$, and suppose that a, b, and c are not zero. Determine a, b, and c.

Solution:

(Due to Justin Shih.) Let r and s be the distinct integral roots that f shares with its derivative. Without loss of generality, assume $r < s$.

Note that r and s must have multiplicity at least 2 or else the derivative would not be zero. On the other hand, r and s cannot have multiplicity greater than 2 because the sum of the multiplicities of all the roots of $f(x)$ cannot exceed 4. Therefore $f(x)$ has exactly 2 roots each with multiplicity 2; we may write

$$f(x) = (x - r)^2(x - s)^2,$$

which has constant term r^2s^2. Computing the derivative, we have

$$\begin{aligned} f'(x) &= 2(x - r)(x - s)^2 + (x - r)^2 2(x - s) \\ &= 2(x - r)(x - s)(2x - r - s) \end{aligned}$$

which has constant term $-2rs(r + s)$. Since f has the form $x^4 + ax^3 + bx^2 + cx - c$, the constant terms of f and f' are $-c$ and c, respectively, and sum to zero. Hence

$$r^2s^2 - 2rs(r + s) = 0.$$

Since $c \neq 0$, we must also have $r, s \neq 0$, so we can divide out by rs to get

$$rs - 2(r + s) = 0$$

Adding to both sides and factoring yields

$$(r - 2)(s - 2) = 4.$$

As r and s are distinct integers, we must have either $r = -2, s = 1$ or $r = 3, s = 6$. Each of these gives a solution.

$$\begin{aligned} f(x) &= (x + 2)^2(x - 1)^2 = x^4 + 2x^3 - 3x^2 - 4x + 4 \qquad \text{and} \\ f(x) &= (x - 3)^2(x - 6)^2 = x^4 - 18x^3 + 117x^2 - 324x + 324 \end{aligned}$$

Thus we find $(a, b, c) = (2, -3, -4)$ or $(a, b, c) = (-18, 117, -324)$.