# Challenge of the Week

March 3–March 9, 2009

## Problem

You have a plate of spaghetti and carry out the following procedure: Pick up two free ends of spaghetti noodles (they may/may not be ends of the same noodle) and tie them together. Repeat until all the ends have been tied. In the end, you'll have loops made of spaghetti.

1. If there were $n$ noodles to begin with, what is the expected number of loops?
2. What's the smallest number of noodles needed to expect more than 5 loops?

## Solution

Let $E(n)$ be the expected number of loops with $n$ noodles. Clearly, with only one noodle on the plate, you will form exactly one loop, so $E(1) = 1$.

Now suppose that $n > 1$. Grab two ends of spaghetti. The chances of forming a loop on the first try is $1/(2n-1)$. If you form aloop set it aside; there will be $n-1$ noodles left. If you didn't form a loop, then you've effectively tied two noodles into one longer noodle, again leaving $n-1$ noodles.

Thus we have $E(n) = 1/(2n-1) + E(n-1)$. Arguing inductively, we have $E(n) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$.

For the second part, we want to find the smallest $n$ so that $E(n) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} > 5$.

Let $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ be the $n$th Harmonic number. The Harmonic numbers have the approximation $H(n) \approx \ln(n) + \gamma$, where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant. The expected number of loops as $E(n) = H(2n) - \frac{1}{2}H(n)$, so a good approximation can be found by solving

$$5 \approx (\ln(2n) + \gamma) - \frac{1}{2}(\ln(n) + \gamma)$$

getting $n \approx \frac{1}{4}e^{10-\gamma} \approx 3091.7$, so a first guess would be 3092 noodles.

Indeed, we can check that $E(3091) = 4.99988$ and $E(3092) = 5.000042$, so we need at least 3092 noodles to expect more than 5 loops.