Challenge Of the Week

March 10—March 16, 2008

Problem

Find all real solutions of the equation $\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}$.

Solution

Start by squaring both sides of the equation and simplifying:

$$x = 4x(1-2x^2)\sqrt{1-x^2}.$$

Clearly $x = 0$ is not a solution, so we can divide by x to get

$$1 = 4(1-2x^2)\sqrt{1-x^2}.$$

Let $y = \sqrt{1-x^2}$. Then changing all x's to y's gives

$$\begin{aligned} 8y^3 - 4y - 1 &= 0 \\ (2y+1)(4y^2 - 2y - 1) &= 0. \end{aligned}$$

Since y must be positive, the only solution to this is $y = \frac{1+\sqrt{5}}{4}$, which gives $x = \pm\frac{\sqrt{10-2\sqrt{5}}}{4}$. For the negative value of x, we note that $4(1-2x^2)\sqrt{1-x^2} < 0$ so cannot be a solution. Thus we find

$$x = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

Variation

Again, square both sides, simplify, and divide by x:

$$1 = 4(1-2x^2)\sqrt{1-x^2}.$$

Now, we make a clever substitution: since x must lie between -1 and 1, we set $x = \sin\theta$, with $-\pi/2 \le \theta \le \pi/2$. (This has the same flavor of doing trig substitutions in integration.) Then we get

$$\begin{aligned} 4(1-2\sin^2\theta)\sqrt{1-\sin^2\theta} &= 1 \\ 4(2\cos^2\theta - 1)\cos\theta &= 1 \\ 8\cos^3\theta - 4\cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(4\cos^2\theta - 2\cos\theta - 1) &= 0. \end{aligned}$$

Since $-\pi/2 \leq \theta \leq \pi/2$, we have $\cos\theta \geq 0$, so the first factor is not a solution. From the second factor we get $\cos\theta = \frac{1+\sqrt{5}}{4}$, so that $\sin\theta = \frac{\pm\sqrt{10-2\sqrt{5}}}{4}$. Since the only positive solution satisfies the original equation, we get

$$x = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$