Challenge of the Week

March 31–April 6, 2009

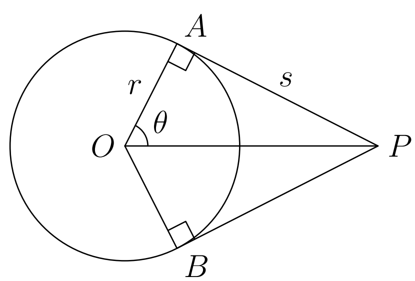
Problem

(Suggested by Gary Raymond.) The Earth (assume a perfect sphere with radius R=3959 miles) is wrapped with a string around the equator. An additional 24π inches of string are added to the string. The string is then lifted at a single point until it is taut. Ignore the effects of gravity. The string is unstretchable.

1. To what height is the string lifted above the earth at the topmost point (in feet)?
2. What is the length of the string not in contact with the earth (in miles)?
3. What is the area between the earth and the string (in square miles)?

Solution

(Due to Justin Shih.) The scenario is depicted in the following diagram:



Let θ be the angle $\angle AOP$, $r = OA = OB = 3959$ miles be the radius of the Earth, and $s = AP = BP$ (also in miles).

The difference between the circumference of the Earth and the length of the string is

$$24\pi \text{ inches} = 2s - 2\theta r = 2r(\tan(\theta) - \theta)$$

Converting both sides to the same units and simplifying gives

$$\frac{2\pi}{2640 \cdot 3959} = \tan(\theta) - \theta.$$

Solving for θ gives $\theta = 0.0076679716098036104$. (No closed-form solution exists here; a numerical answer is the best we can do.)

Knowing θ, we find the height of the string above the earth is

$$\sqrt{r^2 + s^2} - r = 0.11639307407270683 \text{ miles} = 614.555 \text{ feet};$$

the length of the string not in contact with the Earth is

$$2s = 60.71618920362251 \text{ miles;}$$

the area between the string and the Earth is

$$2\frac{rs}{2} - \pi r^2 \frac{2\theta}{2\pi} = r^2(\tan(\theta) - \theta) = 2.3555994915788854 \text{ miles}^2.$$