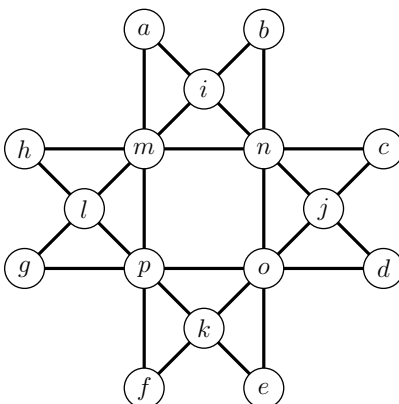


Challenge of the Week

April 7–April 13, 2009

Problem

The figure below is known as a *magic star* if the numbers 1–16 are assigned to the locations a – p in such a way that (1) each number is used exactly once and (2) the numbers along each of the 8 lines have the same sum S .



What are all the possible values for the sum S ?

Solution

Looking along each of the lines, there are 8 relations that must be satisfied:

$$\begin{aligned}
 S &= a + m + p + f \\
 S &= b + n + o + e \\
 S &= h + m + n + c \\
 S &= g + p + o + d \\
 S &= a + i + n + j + d \\
 S &= h + l + p + k + e \\
 S &= b + i + m + l + g \\
 S &= c + j + o + k + f
 \end{aligned} \tag{1}$$

Adding all these up, we find

$$8S = 2(a + b + c + \cdots + o + p) + m + n + o + p. \tag{2}$$

The sum of all the numbers is

$$a + b + c + \cdots + o + p = \frac{16 \cdot 17}{2} = 136. \tag{3}$$

Combining (2) and (3) we get

$$S = 34 + (m + n + o + p)/8,$$

which means that $m + n + o + p$ is a multiple of 8.

If we eliminate $a, b, c, d, e, f,$ and g from the line sums (1) we obtain

$$2(i + j + k + l) = m + n + o + p. \tag{4}$$

The minimum value of $i + j + k + l$ is $1 + 2 + 3 + 4 = 10$, so by equation (4) the minimum value of $m + n + o + p$ is at 20. The first number greater than or equal to 20 and also divisible by 8 is 24; this gives a minimum value for S of 37.

Similarly, the maximum for $m + n + o + p$ is $16 + 15 + 14 + 13 = 58$. The first number less than or equal to 58 which is also divisible by 8 is 56. So the maximum value of S is 41.

So the possible values for S are 37, 38, 39, 40 and 41.

Indeed, all these sums actually work, as shown by the solutions below.

