Challenge of the Week

April 14–April 20, 2009

Problem

Find all polynomials whose coefficients are ± 1 and have all real roots.

Solution

(Based on Koopa's solution.)

Without loss of generality, suppose $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a monic polynomial.

Suppose $p(x)$ has roots $r_1, r_2, \ldots, r_n$ (not necessarily distinct). Then we can write

$$\begin{aligned} p(x) &= \prod_{i=1}^{n}(x - r_i); \qquad \text{multiplying this out gives} \\ &= x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} + \left(\sum_{i<j} r_i r_j\right) x^{n-2} + \cdots + (-1)^n (r_1 r_2 \cdots r_n) \\ &= x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0. \end{aligned}$$

Thus we get relations between the roots of the polynomial and its coefficients. (This is Viete's theorem.) In particular, looking at the coefficients for x^{n-1}, x^{n-2} and the constant term, we get

$$\begin{aligned} \sum_{i=1}^{n} r_i &= -a_{n-1} \\ \sum_{i<j} r_i r_j &= a_{n-2} \\ \prod_{i=1}^{n} r_i &= (-1)^n a_0 \end{aligned}$$

Since each of the coefficients are $a_i = \pm 1$, we have:

$$\sum_{i=1}^{n} r_i^2 = \left(\sum_{i=1}^{n} r_i\right)^2 - 2\left(\sum_{i<j} r_i r_j\right) = (-a_{n-1})^2 - 2a_{n-2} = 1 - 2a_{n-2}.$$

Since the roots r_i are real, we must have $a_{n-2} = -1$, since the sum of squares of real numbers is non-negative. Thus we get

$$\sum_{i=1}^{n} r_i^2 = 3. \qquad (1)$$

We also have

$$\prod_{i=1}^{n} r_i^2 = \left(\prod_{i=1}^{n} r_i\right)^2 = ((-1)^n a_0)^2 = 1. \tag{2}$$

By the Arithmetic Mean-Geometric Mean inequality, we can combine (1) and (2) as:

$$\frac{3}{n} = \frac{1}{n}\sum_{i=1}^{n} r_i^2 \geq \left(\prod_{i=1}^{n} r_i^2\right)^{1/n} = 1.$$

Hence we have $3 \geq n$.

Considering the different cases for $n \leq 3$, we get the following solutions:

1. when $n = 1$, $p(x) = x + 1$ or $x - 1$ (and their negatives)
2. when $n = 2$, $p(x) = x^2 + x - 1$, or $x^2 - x - 1$ (and their negatives)
3. when $n = 3$, since $r_1^2 + r_2^2 + r_3^2 = 3$, we have equality in the AM-GM inequality, meaning that $r_i^2 = r_i$, or $r_i = \pm 1$.

   This implies the three roots are either $1, 1 - 1$ or $-1, -1, 1$ (since the sum of the roots is either 1 or -1). Therefore, $p(x) = (x - 1)^2(x + 1)$ or $(x + 1)^2(x - 1)$ (and their negatives.) Done.