Challenge of the Week

April 21—April 27, 2008

Problem:

A 5×5 square is divided into 25 unit squares. The numbers 1, 2, 3, 4, 5 are inserted into each of the unit squares in such a way that each row, each column, and each of the two main diagonals contains each of the five numbers exactly once. The sum of the four numbers immediately above the diagonal from top-left to bottom-right is called the *value*. What is the highest possible value?

Solution:

The maximum value is 17. This can be shown by explicitly providing an arrangment with value 17, and using a "sudoku-like" argument to show that no higher value is possible.

Claim 1. The value is less than 20.

The only way possible to have a value of 20 is to have four 5's on the superdiagonal:

	5			
		5		
			5	
				5

however such an arrangement makes it impossible to place a 5 on the southeast diagonal. Thus a value of 20 is not achievable; the value is 19 or less.

Claim 2. The value is less than 19.

To have a value of 19, we need three 5's and a 4 on the superdiagonal. There are essentially two ways (after reflections) of doing this:

	4			
		5		
			5	
				5

and

	5			
		4		
			5	
				5

In the first arrangement, we are forced to place a 5 in the top left square, which makes it impossible to have a 5 on the northeast diagonal. In the second arrangement, it is immediately impossible to place a 5 on the southeast diagonal. Thus the maximum value must be 18 or less.

Claim 3. The value is less than 18.

To have a value of 18, the superdiagonal elements must either be 5,5,5,3 or 5,5,4,4. In the first case, the same argument as in claim 2 (with the 4 replaced with a 3) shows this will not work. Thus to have a value of 18 we must have the numbers 5 5 4 4 on the superdiagonal.

After accounting for symmetry, there are four cases:

(1)

	4			
		4		
			5	
				5

To have a 5 on each row, column and main diagonal, we must have:

	4	5		
	5	4		
			5	
				5
5				

at which point it is impossible to place a 4 on the bottom-left to top-right diagonal without having two fours in a row or column.

(2)

	4			
		5		
			4	
				5

To have a 5 on the southeast diagonal, we must have a 5 in the top-left square; this makes it impossible to put a 5 on the northeast diagonal.

(3)

	4			
		5		
			5	
				4

To have a 5 on the southeast diagonal, we must put a 5 in the top-left or bottom-right corner. Reflecting the board, we may assume we place the 5 in the top-left. This forces the arrangement:

5	4			
		5		
			5	
	5			4

and it is impossible to place the last 5.

(4)

	5			
		4		
			4	
				5

The only way to place all the 5's as required is:

	5			
5		4		
		5	4	
				5
			5	

At this point, we must put a 4 in either the top-left or bottom-right corner; from here it is impossible to place all 5 4's.

Thus none of the possible cases for a value of 18 works.

Claim 4. The value 17 is achievable. Here's an example:

3	4	5	2	1
1	5	3	4	2
4	1	2	5	3
2	3	4	1	5
5	2	1	3	4