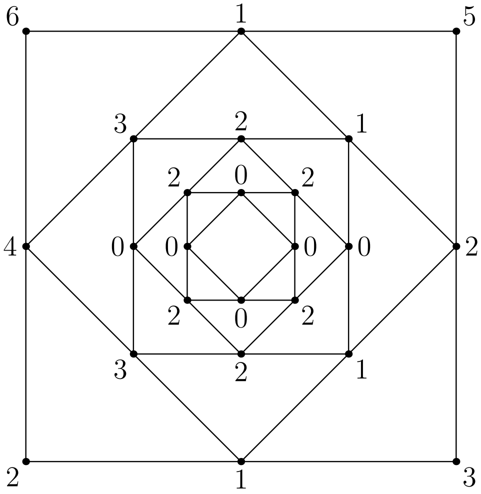
Challenge of the Week

April 7–April 13, 2009

Problem

Draw a square, and label the corners with any four nonnegative integers. At the midpoint of each side, write the difference of the two adjacent numbes, subtracting the smaller one from the larger. This produces a list of four new numbers, written on a smaller square. Repeat this process.

For example, starting with the numbers $2, 3, 5, 6$, we generate the following squares:



It's not really necessary to draw the squares; we can just write the iterations as a table. Given the row (a, b, c, d), the next row is $(|a-b|, |b-c|, |c-d|, |d-a|)$:

2	3	5	6
1	2	1	4
1	1	3	3
0	2	0	2
2	2	2	2
0	0	0	0

Notice that we reached a row of zeros (equivalently, a square with all its vertices labeled 0) in 6 steps.

The problem: Prove that the process always reaches (0 0 0 0) eventually.

(This is a tricky problem, but there are lots of other interesting questions to think about along the way: how can you pick starting numbers so that many steps are needed to get to $(0, 0, 0, 0)$? Could you pick starting values so it takes 100 steps? What happens when non-integer values are used—how long does the sequence $(0, 1, 6, \pi)$ take? What happens if we start with pentagons or hexagons instead of squares?)

Solution

Here's a clever way to show that the rows must eventually go to (0, 0, 0, 0). We need a quick lemma first:

Lemma: Given any row, the process yields a row with all even numbers after at most 4 additional steps.

Proof: There are only a few possible cases for even/odd-ness for a given row; we can just try each case to see what happens. We can reduce the number of cases to check because of the symmetries of the squares. Write "e" to denote an even number and "o" for an odd number.

Starting Row	Iterations
(o, o, o, o)	$\rightarrow (e, e, e, e)$
(o, o, o, e)	$\rightarrow (e, e, o, o) \rightarrow (e, o, e, o) \rightarrow (o, o, o, o) \rightarrow (e, e, e, e)$
(o, o, e, e)	$\rightarrow (e, o, e, o) \rightarrow (o, o, o, o) \rightarrow (e, e, e, e)$
(o, e, o, e)	$\rightarrow (o, o, o, o) \rightarrow (e, e, e, e)$
(o, e, e, e)	$\rightarrow (o, e, e, o) \rightarrow (o, e, o, e) \rightarrow (o, o, o, o) \rightarrow (e, e, e, e)$
(e, e, e, e)	

□

Now for the main claim.

Theorem: The process yields rows of zeros after a finite number of steps.

Proof: Given a row $R = (a, b, c, d)$, we compute the next row as $R' = (|a-b|, |b-c|, |c-d|, |d-a|)$.

Define the *size* of the row R to be $size(R) = \max\{a, b, c, d\}$. Since the computation of R' only uses subtraction, we have $\max\{a, b, c, d\} \geq \max\{|a-b|, |b-c|, |c-d|, |d-a|\}$ so that $size(R) \geq size(R')$. Thus as we compute future rows, the size can only decrease:

$$size(R) \geq size(R') \geq size(R'') \geq \cdots$$

To start, run the process for a few steps until reaching an all-even row S (such a row exists because of the lemma.) We can write the row as $S = (2w, 2x, 2y, 2z)$ for some whole numbers w, x, y, z. Let $T = (w, x, y, z)$. Observe that if we run the process from S we will reach a row of zeros if and only if we reach a row of zeros starting from T. Note also that if $size(S) > 0$, then $size(T) = \frac{1}{2} size(S)$ is strictly smaller than $size(S)$.

Continue the process starting from T. Each time we reach an all even row, we can factor out another 2. If a row of zeros never appears, then the maximal entries of the factored rows provide a decreasing list of positive integers that never ends. Since this is impossible, there must be a row of zeros eventually. □

A lot of research has been done on this puzzle. See `http://mathed.uta.edu/~kribs/diffy.html` for a long list of references.