Challenge Of the Week

May 12—May 18, 2008

Problem:

Find all integer solutions to $3x^2 + 1 = 4y^3$.

Solution:

The shortest way to do this is to rewrite the equation the following way:

$$(1+x)^3 + (1-x)^3 = (2y)^3$$

Using Fermat's Last Theorem[1] this equation has no nontrivial solutions, so we must have one of the three terms be 0.

- If the first term is 0, then we have $x = -1$, forcing $y = 1$.
- If the second term is 0, then we have $x = 1$, forcing $y = 1$.
- If the last term is 0, then we have $(1+x)^3 + (1-x)^3 = 3x^2 + 1 = 0$, which has no real solutions.

So the only solutions are $(x, y) = (-1, 1)$ and $(1, 1)$.

[1] As Travis Willse points out, quoting Fermat's Last theorem leads to a fast proof, though it's inefficient mathematically. The degree 3 case of the theorem can be proven using elementary methods; see *An Introduction to the Theory of Numbers* by Hardy and Wright.