Challenge Of the Week

May 12—May 18, 2008

Problem:

A man buys two boxes of matches and puts them in his pocket. Every time he has to light a match, he selects at random one box or the other. After some time, he takes one of the boxes from his pocket, and finds it is empty (after absentmindedly placing the box back in his pocket after using the last match). Supposing that each box originally had n matches, what is the probability that there are now k matches in the other box? (Here, $0 \leq k \leq n$.)

Solution:

Let us call the empty box A and the other box B. We want to find the probability that when the man tries to take an $(n+1)$-st match from box A, exactly k matches remain in B. For this to happen the man must have drawn a total of $2n-k$ matches, $n-k$ from box B and n from box A.

What's the probability of this happening? There are 2^{2n-k} equally likely outcomes for choosing the boxes for $2n-k$ matches. Of these, number of ways the man could have taken exactly n matches are from A, is "$2n-k$ choose n", or $\binom{2n-k}{n} = \frac{(2n-k)!}{n!(n-k)!}$. Hence the desired probablity is:

$$P = \frac{\binom{2n-k}{n}}{2^{2n-k}}$$