Challenge Of the Week

June 2—June 8, 2008

Problem:

Two players, Alphonse and Beatrice, take turns removing marbles from a jar initially containing 500 marbles. The player who takes the last marble wins. The catch is that on each turn, the number of marbles that is withdrawn must be a power of two. Is there a winning strategy for either of the players?

Solution:

If the number of marbles remaining is not a multiple of 3, then the first player can win, otherwise the second player can win. In this case, with 500 marbles, Alphonse can win by taking 2, 8, 32, or 128 marbles.

The easiest way to analyze this game is to work backwards starting with a small number of marbles. Clearly Alphonse will win if there are 1 or 2 marbles. However, if there are 3 marbles, then Alphonse must take 1 or 2, allowing Beatrice to win. If there are 4 or 5 marbles, then Alphonse can take 1 or 2 to reach a winning position with 3 marbles. With 6 marbles, Alphonse cannot win since he cannot force Beatrice to have 0 or 3 marbles. Beatrice to have 3 marbles. Continuing in this manner suggests that the first player can win exactly when the number of marbles is not a multiple of 3. We now show this.

Suppose the number of marbles n is not a multiple of 3. If $n = 3k+1$, then Alphonse can take 1 marble (or generally 2^{2m} marbles) to leave Beatrice with a multiple of three marbles. If $n = 3k+2$, then Alphonse can take 2 marbles (or generally 2^{2m+1} marbles) to leave Beatrice with a multiple of three marbles.

When Beatrice is left with a multiple of 3 marbles, she cannot win immediately on her turn since no power of two is a multiple of 3. After her turn, Alphonse will again have a non-multiple-of-3 number of marbles, with fewer marbles total. Thus the game must stop eventually, and since Beatrice never wins on any of her turns, Alphonse must win.

Had we started with a multiple of 3 marbles, then the second player has a win: any move Alphonse makes reduces the number to some non-multiple-of-3, and Beatrice can then force a win using Alphonse's strategy above.