Challenge of the Week

June 23–July 6, 2008

Problem

Find all the integer solutions to the system of equations

\begin{align*}
ab + cd &= -1 \quad (1) \\
ac + bd &= -1 \quad (2) \\
ad + bc &= -1 \quad (3)
\end{align*}

Solution

If we subtract (3) from the (2), we get \((a - b)(c - d) = 0\), so that \(a = b\) or \(c = d\). Similarly, subtracting (3) from (1) gives \((a - c)(b - d) = 0\) so that \(a = c\) or \(b = d\). Considering the different combinations of these equalities, we must have one of the following possibilities:

- \(a = b = c; d\) different
- \(a = b = d; c\) different
- \(a = c = d; b\) different
- \(b = c = d; a\) different
- \(a = b = c = d\)

If all four variables are equal, then the first equation becomes \(2a^2 = -1\), which is impossible. Thus we must have three variables equal and one different; that is, the solutions have one of the forms:

\((x, x, x, y)\), \((x, x, y, x)\), \((x, y, x, x)\), \((y, x, x, x)\)

From the symmetry of the equations, if \((a, b, c, d)\) is a solution, then so are each of the permutations \((a, b, d, c), (a, c, b, d), \ldots\). So without loss of generality, consider a solution of the form \((a, b, c, d) = (x, x, x, y)\); then equation (1) becomes

\[x^2 + xy = -1.\]

Solving for \(x\) gives

\[x = \frac{y \pm \sqrt{y^2 - 4}}{2}.\]

Since \(x\) is an integer, \(y^2 - 4\) must be a square number, so that \(y^2 - 4\) and \(y^2\) are both squares. However, the only square numbers that differ by 4 are 0 and 4, thus \(y^2 = 4\). This means that \(y = \pm 2\). Plugging in for \(x\) gives \(x = y/2 = \mp 1\). Thus we find the solutions \((-1, -1, -1, 2)\) and \((1, 1, 1, -2)\), and their permutations.