Challenge Of the Week
October 13—October 19, 2009

Problem:

Five friends host a wine party. At the end, they are left with 45 bottles of wine, of which 9 each are full, three-quarters full, half-full, one-quarter full, and empty.

The friends wish to divide the remaining wine and the bottles, without transferring wine into different bottles, so that (1) each receives the same amount of wine (2) each receives the same number of bottles (3) each receive at least one of each kind of bottle, and (4) no two of them receive the same number of every kind of bottle.

Solution:

While this puzzle lends itself to brute-force checking, we can simplify the problem somewhat to minimize our work. To start, measure quantities of wine in quarter bottles (qb) to avoid dealing with fractions. Thus we can refer to the different types of bottles with numbers 0 (an empty bottle) to 4 (a full bottle).

Since each person must receive at least one of each kind of bottle, start by giving one of each bottle to everyone (that’s 5 bottles × 5 friends). Note that in doing this, everyone receives the same number of bottles and the same quantity of wine. Thus the problem reduces to the following:

There are 20 bottles; 4 of each type. We want to distribute them to five friends so that (1) they receive the same amount of wine, (2) they receive the same number of bottles, and (3) two receive the same number of every kind of bottle.

The total amount of wine is $4 \times (0 + 1 + 2 + 3 + 4) \text{ qb} = 40 \text{ qb}$; thus each person must receive $40/5 = 8 \text{ qb}$ of wine. Each person must get $20/5 = 4$ bottles. There are only a few ways to get 8 qb of wine using 4 bottles:

$$
\begin{align*}
8 &= 4 + 4 + 0 + 0 \\
8 &= 4 + 3 + 1 + 0 \\
8 &= 4 + 2 + 2 + 0 \\
8 &= 4 + 2 + 1 + 1 \\
8 &= 3 + 3 + 2 + 0 \\
8 &= 3 + 3 + 1 + 1 \\
8 &= 3 + 2 + 2 + 1 \\
8 &= 2 + 2 + 2 + 2
\end{align*}
$$

Now, trying combinations of these so that everyone gets different amounts, we find three essentially different solutions. Each row in the table below indicates the bottle types that
a person should get. For example, in the first solution, person “A” receives two full bottles and two empty bottles, denoted by “4 4 0 0.”

<table>
<thead>
<tr>
<th>person</th>
<th>solution 1</th>
<th>solution 2</th>
<th>solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4 4 0 0</td>
<td>4 4 0 0</td>
<td>4 4 0 0</td>
</tr>
<tr>
<td>B</td>
<td>4 3 1 0</td>
<td>4 3 1 0</td>
<td>4 2 2 0</td>
</tr>
<tr>
<td>C</td>
<td>4 2 2 0</td>
<td>4 2 1 1</td>
<td>4 2 1 1</td>
</tr>
<tr>
<td>D</td>
<td>3 3 1 1</td>
<td>3 3 2 0</td>
<td>3 3 2 0</td>
</tr>
<tr>
<td>E</td>
<td>3 2 2 1</td>
<td>3 2 2 1</td>
<td>3 3 1 1</td>
</tr>
</tbody>
</table>

To recover the solutions to the original problem, give one additional bottle of each type to each person.