Challenge Of the Week

October 27—November 2, 2008

Problem:

Solve the following cryptarithm. Each letter D, I, B, O, T, U, R, and N stands for a different digit 0–9.

\[
BOB/DID = .TURNTURNTURNTURN\ldots
\]

Solution:

(Based on Lloyd Sakazaki’s solution.) There are two answers to this:

\[
212/606 = .3498\ldots \quad \text{and} \quad 242/303 = .7986\ldots
\]

In the end, this problem reduces to checking a fair number of cases. However, we can simplify the problem somewhat to minimize this. First get rid of the repeating decimal. Subtract

\[
10000 \frac{BOB}{DID} = \frac{TURNTURNTURNTURN\ldots}{-} \frac{BOB}{DID} = \frac{.TURNTURNTURNTURN\ldots}{9999 \frac{BOB}{DID} = \frac{TURNTURNTURNTURN\ldots}{}}
\]

Multiplying both sides by DID, we get the simpler problem 9999 \cdot BOB = DID \cdot TURN.

Notice that 9999 factors as 3^2 \cdot 11 \cdot 101; so the right hand side must be divisible by 9, 11, and 101. Furthermore, 101 cannot be a divisor of TURN, because every four digit multiple of 101 has the form ABAB. (That is, AB \cdot 101 = ABAB.) Thus DID = D \cdot 101, and we find that I = 0.

We can now divide the equation by 101 to get

\[
99 \cdot BOB = D \cdot TURN
\]

We know that B < D, otherwise the original fraction would be greater than 1. (This means that D is 2 or larger.) From here, we focus on the units digit of the above equation; equivalently consider the equation modulo 10:

\[
9 \cdot B = D \cdot N \pmod{10}
\]
Considering the possible values for D in turn, each one yields only a few valid possibilities for B and N. In the end, we get 14 possibilities:

\[
\begin{align*}
D &= 3, \quad B &= 2, \quad N = 6 \\
D &= 4, \quad B &= 2, \quad N = 7 \\
D &= 6, \quad B &= 2, \quad N = 3 \\
D &= 6, \quad B &= 2, \quad N = 8 \\
D &= 6, \quad B &= 4, \quad N = 1 \\
D &= 7, \quad B &= 2, \quad N = 4 \\
D &= 7, \quad B &= 3, \quad N = 1 \\
D &= 7, \quad B &= 4, \quad N = 8 \\
D &= 7, \quad B &= 6, \quad N = 2 \\
D &= 8, \quad B &= 2, \quad N = 1 \\
D &= 8, \quad B &= 2, \quad N = 6 \\
D &= 8, \quad B &= 4, \quad N = 2 \\
D &= 8, \quad B &= 4, \quad N = 7 \\
D &= 8, \quad B &= 6, \quad N = 3
\end{align*}
\]

Trying each of these combinations in the original problem, it is fairly quick to determine that the combination is either impossible, or to determine the remaining values O, T, U, and R.