Challenge of the Week

December 1–December 6, 2009

Problem

Suppose we have a circle with $2n$ segments, $n$ of them colored red and $n$ colored black in some random order. Make a smaller concentric circle, also with $2n$ segments, again with $n$ colored red and $n$ colored black, again in some random order (not necessarily the same order as the larger circle). We can spin the smaller circle relative to the larger circle so that segments line up. Show that there is some way to spin the smaller circle so that at least half (that is, $n$) of the segments have matching colors.

Solution

Let’s count the number of matches we have on average. To do this, add up the total number of matches over every possible spin, and divide by the number of spins.

Consider one red sector in the small circle; as the circle is spun, it will match each of the $n$ of the outer-circle reds in turn. Similarly, every black sector will match the outer $n$ blacks. So as we spin, the total number of matches is

$$n \text{ matches} \cdot n \text{ red segments} + n \text{ matches} \cdot n \text{ black segments} = 2n^2$$

Dividing by the number of rotations $2n$, the average number of matches is $2n^2/(2n) = n$.

Since the average number of matches is $n$, at least one of the rotations must have at least $n$ matches. (If all rotations had fewer than $n$ matches then the average number of matches would have to be less than $n$, a contradiction.)