Challenge of the Week

December 9, 2009–January 4, 2010

Problem

Jason has a 4 foot by 8 foot rectangular piece of plywood that he would like to make into the top of a circular table.

- What is the largest table that he can make by cutting out two identical semicircles (and throwing the remaining wood away)? Hint: it’s possible to make a table with radius bigger than 8/3.
- Instead of cutting out semicircles, suppose Jason can cut two identical pieces of any shape to make a circular table. What’s the largest table he can make? (And how should he do it?)
- Finally, suppose there’s no restriction on the shape of the pieces. What’s the largest table possible with two pieces of any shape?

Solution for two semicircles

The best arrangement looks like this:

Notice that there’s no slack in this arrangement—each semicircle is touching 3 sides of the rectangle and the other semicircle. So this solution should be at least locally optimal.

To find the exact dimensions, suppose that the semicircle has radius \( r \) and touches the top of the rectangle at \( (\alpha, 4) \). Since the semicircle is flush with the bottom and left, the center is at \( (r, r) \).

The distance from the point \( (r, r) \) to the point \( (\alpha, 4) \) is \( r \), so we get the equation

\[
(r - \alpha)^2 + (r - 4)^2 = r^2. \quad (1)
\]
The line that cuts across the board goes through the points \((r, r)\) and \((4, 2)\), so is described by

\[ y - r = \frac{r - 2}{r - 4}(x - r) \quad (2) \]

Since \((\alpha, 4)\) is on the line, it satisfies (2), and we get a second equation:

\[ 4 - r = \frac{r - 2}{r - 4}(\alpha - r) \quad (3) \]

 Eliminating \(\alpha\) from the two equations (1) and (3) gives a 4th degree polynomial for \(r\):

\[ r^4 - 24r^3 + 144r^2 - 352r + 320 = 0 \]

Numerically solving this for \(r\) gives: \(r \approx 2.7054458, \alpha \approx 0.3298266374\).

Solution for two identical pieces

Here’s the best arrangement I know of:

To form the pieces, we cut a zig-zag from the point \((a, 4)\) on the top of the rectangle, cut to a horizontal segment of length \(2b\) in the middle, and then cut symmetrically to the point \((8 - a, 0)\) on the bottom of the rectangle.

Let \(O\) and \(P\) be midpoints of the segments they lie on; they have coordinates:

\[ O = (O_x, O_y) = \left( \frac{12 - a + b}{2}, 1 \right) \]
\[ P = (P_x, P_y) = \left( \frac{4 + a - b}{2}, 3 \right). \]

We want to find \(a\) and \(b\) so that the pieces are tight on the left and right sides of the rectangle, and also so that \(O\) and \(P\) are
1. at the center of the circle after rearrangement, and
2. at points of tangency when the pieces are arranged in the rectangle.

Supposing that the circle has radius \( r \), the distance from \( O \) to right side is

\[
\frac{r = 8 - \frac{12 - a + b}{2}}{} \tag{4}
\]

Viewing \( O \) as the center of the circle and \( P \) as the edge of the circle on the piece on the right, we see the distance from \( O \) to \( P \) is also \( r \). Thus we have:

\[
r^2 = (O_x - P_x)^2 + (O_y - P_y)^2 \tag{5}
\]

Finally, if \( P \) is a point of tangency, then the slope of the circle must match the slope of the line segment. This gives the relation

\[
\frac{2}{a + b - 4} = -\frac{12 - a + b}{1 - P_x}
\]

We can solve (4), (5), and (6) to get

\[
\begin{align*}
\frac{2}{a + b - 4} &= \frac{12 - a + b}{1 - P_x} \\
\frac{2}{a + b - 4} &= \frac{12 - a + b}{1 - P_x} \\
r &= \frac{2}{3} \left( 8 - \sqrt{13} \right) \\
a &= \frac{1}{12} \left( 20 + 6 \left( 8 - \sqrt{13} \right) \right) \\
b &= \frac{1}{12} \left( 68 - 10 \left( 8 - \sqrt{13} \right) \right)
\end{align*}
\]

This gives \( r \approx 2.92963 \).

**Solution for any two pieces**

As far as I know, it’s not possible to improve on the solution above with two identical pieces. The obvious thing to try is to cut the circle with a single chord, and arrange the pieces like this:
The following analysis is adapted from Lloyd Sakazaki’s solution.

The larger piece has center at \((r, r)\). Points on the circular boundary touch the rectangle at \((a, 4)\) and \((2r - a, 4)\) on the top side of rectangle, \((0, r)\) on the left side of the rectangle, and \((r, 0)\) on the bottom side of the rectangle.

The smaller piece has center at \((8 - r, r)\). Points on the circular boundary touch the rectangle at \((8 - a, 4)\) on the top side of rectangle, \((8, r)\) on the right side of the rectangle, and \((8 - r, 0)\) on the bottom side of the rectangle.

Since \((a, 4)\) sits on the circle centered at \((r, r)\), we have \((a - r)^2 + (4 - r)^2 = r^2\), which gives

\[ a = r - 2\sqrt{2r - 4}. \]  

(7)

Observe that the chord between the two points where this circle touches the top of the rectangle is \(\ell = 2r - a - a = 2r - 2a\).

Next, label as \((b, c)\) the coordinates of the lower chord-endpoint on the smaller piece. There are two conditions that this point must satisfy:

- The distance from \((b, c)\) to the point \((8 - r, r)\) must be the radius:
  \[ (b - 8 + r)^2 + (c - r)^2 = r^2 \]  
  (8)

- The length of the chord between the points \((8 - a, 4)\) and \((b, c)\) must be \(\ell\).
  \[ (8 - a - b)^2 + (4 - c)^2 = (2r - 2a)^2 \]  
  (9)

Finally, we require that the larger and smaller pieces touch at a single point, which is equivalent to stipulating that \(r\) is the distance from the center of the larger piece, \((r, r)\), to the chord line of the smaller piece, whose equation is \(y - c = s(x - b)\), where the slope is

\[ s = (4 - c)/(8 - a - b). \]  

(10)

Using the standard result for the distance from a point to a line allows us to write

\[ r = \frac{|(s - 1)r - (sb - c)|}{\sqrt{s^2 + 1}}. \]  

(11)

Numerically solving equations (8), (9), (10), and (11) gives \(r \approx 2.850\).