Problem:

Show that there is no set of positive integers \( \{a, b, c, d\} \) that satisfy the conditions \( a^3 = b^5 \), \( c^3 = d^5 \), and \( c - a = 32 \).

Solution:

Consider the quantity \( x = a^3 = b^5 \). We see \( x \) is a cube and a 5th power, so must be a 15th power. That is, \( x = m^{15} \) for some positive integer \( m \). In particular, this means that \( a^3 = m^{15} \), so \( a = m^5 \). We conclude that \( a \) is a 5th power. Using the same reasoning on the quantity \( y = c^3 = d^5 \), we can write \( c \) as a 5th power; \( c = n^5 \) for some \( n \).

The final condition is \( c - a = 32 \); this translates to \( n^5 - m^5 = 32 \). We want to show no solutions to \( n^5 - m^5 = 32 \) exist in positive integers.

At this point, we can rearrange this to \( n^5 = 2^5 + m^5 \) and then observe that no solution is possible by Fermat’s Last Theorem. However, we give an alternate, elementary argument.

- If \( n = 2 \), then the only smaller integer is \( m = 1 \), but we see that \( 2^5 - 1^5 \neq 32 \), so this isn’t a solution.
- If \( n > 2 \), then we compute

\[
\begin{align*}
 n^5 - m^5 &= (n - m)(n^4 + n^3m + n^2m^2 + nm^3 + m^4) \\
 &\geq (n - m)(3^4 + 3^3m + 3^2m^2 + 3m^3 + m^4) \\
 &\geq 3^4 \\
 &\neq 32
\end{align*}
\]

Thus no set of integers \( \{a, b, c, d\} \) satisfy the conditions.