Challenge Of the Week
April 20 – April 26, 2010

Problem:

Find all the functions \( f : \mathbb{R} \setminus \{-1, 0, 1\} \to \mathbb{R} \) that satisfy the following equation

\[
(f(x))^2 \cdot f\left(\frac{1 - x}{1 + x}\right) = 64x
\]  

(1)

Solution:

First, notice that the map given by \( y = \frac{1 - x}{1 + x} \) has an inverse given by \( x = \frac{1 - y}{1 + y} \), that is, it is its own inverse as a map on \( \mathbb{R} \setminus \{-1\} \). Using this value for \( x \) in the given equation we obtain

\[
\left[ f\left(\frac{1 - y}{1 + y}\right)\right]^2 \cdot f(y) = 64 \frac{1 - y}{1 + y}
\]  

(2)

Squaring equation (1) at \( x = y \), yields

\[
\left[ f\left(\frac{1 - y}{1 + y}\right)\right]^2 \cdot (f(y))^4 = 64^2 y^2.
\]  

(3)

Now, from equation (1), we see that the only possible zeroes of \( f \) are at 0 and 1, so it is safe to assume that \( f \) is never zero. Dividing, then, (3) and (2)

\[
(f(y))^3 = 64y^2 \frac{1 + y}{1 - y},
\]  

(4)

Which shows that there is at most one such \( f \), given by

\[
f(x) = 4 \sqrt[3]{x^2 \frac{1 + x}{1 - x}}.
\]  

(5)

One the other hand, it is direct to verify that this function satisfies equation (1).