Challenge Of the Week

March 4 – March 10, 2010

Problem:

In a theater, 50 people watch a very boring play. The play was so boring that every single person fell asleep once (only once!). At the end of the play, two actors make the following statements:

Actor 1 During the play, there was at least one moment in which eight or more people were asleep simultaneously.

Actor 2 In the theater, there is a group of at least 8 people in which no pair of people slept simultaneously.

Although we can’t say which actor is telling the truth, we do know that at least one of them is. Why?

Solution:

Order the 50 people in the audience by when they felt asleep: person 1 felt asleep first, person 2 second, and so on. If two people felt asleep at the exact same time, choose one of them to be labeled before the other one.

Assume that Actor 1 is lying. Then, no group of eight people can be asleep at the same time, which means that somebody among 1,2,3,4,5,6,7 has to wake up before 8 falls asleep. Let that person be $a_1$. By a similar reasoning, somebody among 8,9,10,11,12,13,14 has to wake up before 15 falls asleep. Call this person $a_2$. Working through this we have that in the group $a_1, a_2, \ldots, a_7$, and 50, no pair of people is asleep at the same time (because each of them wakes up before the next one falls asleep) and Actor 2 is telling the truth.

So, it is impossible that both of them lie, and therefore one of them is telling the truth.

Outstanding solutions

By Congpa You: If Actor 1 is not true, then we can assume there are 7 beds for those who are sleeping. One has to wait until a bed is available before he can fall asleep in that bed. If Actor 2 is not true, then each bed can only be slept by 7 different people. So the most people can sleep is $7 \times 7 = 49$. So for 50 people all fell asleep once, at least one actor is true.
By Jason Shaw: If Actor 1 is telling the truth, then we are done. Suppose that Actor 1 is lying. Then throughout the play, no more than 7 people are ever asleep at once. When each person falls asleep, assign them a positive integer according to the following constraints:

1) They are not assigned an integer used by anyone else sleeping at the time
2) They are assigned the lowest unused integer

Then by hypothesis, no person is assigned a number higher than 7: when a person falls asleep, no more than 6 people were asleep already, so at least one of 1,2,...,7 is unused. Each of 50 people is assigned one of 7 integers, so by the pigeonhole principle, at least one integer n is used by ceiling(50/7) = 8 people. Since no sleeper was assigned an integer used by someone sleeping at the same time, the people who used n constitute a group of at least 8 in which no pair slept simultaneously. Thus Actor 2 is telling the truth.

If Actor 1 is telling the truth, the claim is upheld. If Actor 1 is lying, then Actor 2 is telling the truth, and the claim is upheld. Therefore the claim is true.