Challenge Of the Week

March 18—March 24, 2008

Problem:

Let f be a function such that $f(n)$ is an integer whenever n is an integer. Prove that there is some integer n so that $f(f(n)) \neq n+3$.

Solution:

We argue by contradiction. Suppose that $f(f(n)) = n+3$ for all integers n. Then

$$f(n+3) = f(f(f(n))) = f(n)+3$$

for all n. Now define $g(n) = f(n) - n$, so that $f(n) = n + g(n)$. Then

$$g(n+3) = f(n+3) - (n+3) = f(n) - n = g(n)$$

so g is periodic (with period 3) on the integers. That is, we can write

$$g(n) = \begin{cases} a & \text{if } n \equiv 0 \pmod 3 \\ b & \text{if } n \equiv 1 \pmod 3 \\ c & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

for some integers a, b and c. We can correspondingly write f as

$$f(n) = \begin{cases} n+a & \text{if } n \equiv 0 \pmod 3 \\ n+b & \text{if } n \equiv 1 \pmod 3 \\ n+c & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

We now consider each of the possible residues for $a \pmod 3$:

1. Suppose $a \equiv 0 \pmod 3$. Observe that $f(0) = a$. Then we get $3 = f(f(0)) = f(a) = a + a$, so that $2a = 3$, and a is not an integer, as required.
2. Suppose $a \equiv 1 \pmod 3$. Then $3 = f(f(0)) = a + b$, and so $b \equiv 2 \pmod 3$

   Now consider the possible remainders for c:

   (a) Suppose $c \equiv 0 \pmod 3$. Then $5 = f(f(2)) = f(2+c) = 2+c+c$, and we get $3 = 2c$, meaning c is not an integer, which is impossible.

(b) Suppose $c \equiv 1 \pmod 3$. Then $f(2) = 2 + c \equiv 0 \pmod 3$, and so $5 = f(f(2)) = 2 + c + a$ and thus $c + a = 3$. But this is false since both $a, c \equiv 1 \pmod 3$.

(c) Suppose $c \equiv 2 \pmod 3$. Then $f(2) = 2 + c \equiv 1 \pmod 3$, and so $5 = f(f(2)) = 2+c+b$. We deduce that $c+b \equiv 0 \pmod 3$, which is wrong, since $b, c \equiv 2 \pmod 3$.

3. Suppose $a \equiv 2 \pmod 3$. This argument is essentially the same as for case $a \equiv 1 \pmod 3$.

In every case, we reach a contradiction. Thus there must be some n so that $f(f(n)) \neq n+3$.