# WHITTAKER MODELS, NILPOTENT ORBITS AND THE ASYMPTOTICS OF HARISH-CHANDRA MODULES

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Abstract. We study the existence of Whittaker models for Harish-Chandra modules. In a real rank two setting, we prove Matumoto's conjecture, establishing the equivalence of a nilpotent orbit condition, the existence of a Whittaker model and an asymptotic condition; the equivalence of these three conditions fails in higher rank.

## 1. Introduction.

One of the most important theorems in the representation theory of a semisimple Lie group is the *Subrepresentation Theorem:* Every irreducible admissible representation can be realized as an invariant subspace of some principal series representation. Using the theory of matrix coefficient asymptotics, one can give an elegant account that such embeddings must exist, but a complete determination of all embeddings is still mysterious and unknown. For certain problems, knowing all possible embeddings is not important. For example, in order to classify the irreducible admissible representations (i.e. *Langlands Classification*), the embeddings one must understand are easily determined; in part, this is due to the fact that these embeddings are "maximal" among the set of all such embeddings. However, when studying embeddings into more general types of induced modules (e.g. the existence of Whittaker models), the non-maximal embeddings into principal series representations are of crucial importance. In this article, we locate embeddings of an opposite character from the maximal embeddings of Langlands classification; what one might refer to as "minimal embeddings". These are the most difficult embeddings to understand and, in general, there is no known procedure to compute them.

Our motivation is a conjecture of H. Matumoto [26] and his subsequent work [27], [28]. Simply put, the conjecture links three *a priori* different notions: the singularity theory of irreducible Harish-Chandra modules (as encoded in the associated variety of the annihilator), the theory of matrix coefficient asymptotics (as encoded by the Jacquet module), and the existence of embeddings into particular induced representations (referred to as Whittaker models). From one perspective, the conjecture implies the existence of very special "minimal embeddings" of representations into principal series representations; these minimal embeddings, when combined with prior work of Matumoto and Goodman-Wallach, yield Whittaker models. So, our ability to exhibit the right kind of minimal embeddings into principal series amounts to an existence theorem for Whittaker models; this is perhaps

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the most important consequence of this paper. However, in another light, one can view our results as an attempt to revisit and reinterpret the authors joint program with L. Casian in [8]-[10]. Whereas, the former program focused on the $\mathfrak{g}$-structure of Jacquet modules, the ideas in this paper advance the philosophy of describing “nice submodules” of Jacquet modules via a connection with the theory of nilpotent orbits. From this vantage point, adopting the Hecke module framework of [8]-[10], we are studying a delicate relationship between double cell Weyl group representations in the Harish-Chandra module setting and right cell Weyl group representations in a highest weight module setting. In the real rank two Hermitian symmetric case, we will prove Matumoto’s conjecture is true. A detailed analysis in $Sp_6\mathbb{R}$ shows the conjecture fails, in general, for higher real rank. In addition, we will indicate the conjecture is “almost” true for the general real rank two case.

As usual, more precision requires much more notation and terminology. We fix $G$ to be a connected semisimple real matrix group and $P_m = M_m A_m N_m \subset G$ a minimal parabolic subgroup compatible with an Iwasawa decomposition $G = KA_m N_m$. We denote real Lie algebras by the notation $\mathfrak{g}_o, \mathfrak{k}_o$, etc., their complexifications without the subscript “o”. Fix an *Iwasawa Borel subalgebra* $\mathfrak{b} \subset \mathfrak{p}_m$, which induces a Bruhat ordering on the full Weyl group $W$; we choose the ordering so that $e$ (resp. $w_o$) is the unique minimal (resp. maximal) element. We will be working primarily in one of two types of categories of representations; each setting requires some notation, all of which is standard and reviewed in §2. Specifically, we work within the category of Harish-Chandra modules $\mathcal{HC}_o$ with the same infinitesimal character as a fixed finite dimensional representation $F$ of $G$. The irreducible and standard modules in this category are parametrized by a finite partially ordered set $\mathcal{D}$; if $\delta \in \mathcal{D}$, then $\bar{\pi}(\delta)$ and $\pi(\delta)$ denote the irreducible and standard modules, respectively. In addition, if $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then we recall the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ of highest weight modules. In this case, the set of minimal length right coset representatives $W^P$ is a parameter set for the irreducible modules $L_{\mathfrak{p}}(w)$ and the generalized Verma modules $N_{\mathfrak{p}}(w)$; our conventions are setup so that $N_{\mathfrak{p}}(e) = L_{\mathfrak{p}}(e)$; see §2 for more details.

It is important to recall the assignment $V \rightsquigarrow \mathcal{O}_V$, which associates to each irreducible $\mathfrak{U}(\mathfrak{g})$-module $V$ a nilpotent orbit $\mathcal{O}_V$ in $\mathfrak{g}^*$ (or $\mathfrak{g}$). This requires that we begin with the annihilator $I_V$ of $V$ in $\mathfrak{U}(\mathfrak{g})$; any such ideal is called a *primitive ideal*, by definition. The associated graded object $\mathrm{gr}I_V$ is a graded ideal in $\mathrm{gr}\mathfrak{U}(\mathfrak{g}) \cong S(\mathfrak{g})$. As such, it has an associated variety $\mathcal{V}(\mathrm{gr}I_V)$ of common zeros in $\mathfrak{g}^*$. Since $I_V$ is graded (resp. $G_{ad}$-stable), this variety is a cone in $\mathfrak{g}^*$ (resp. is $G_{ad}$-stable). The ideal $I_V$ meets the center $\mathfrak{Z}(\mathfrak{g})$ in an ideal of codimension one and since the associated graded algebra of $\mathfrak{Z}(\mathfrak{g})$ identifies with the space $S(\mathfrak{g})^{G_{ad}}$ of $G_{ad}$-invariant polynomials in $S(\mathfrak{g})$, it follows that $\mathrm{gr}I_V$ meets $\mathrm{gr}\mathfrak{Z}(\mathfrak{g})$ in its augmentation ideal, consisting of all $G_{ad}$-invariant polynomials with zero constant term. Making appropriate identifications, this implies that $\mathcal{V}(\mathrm{gr}I_V)$ sits inside the nilcone

$$\mathcal{N} = \{X \in \mathfrak{g} \,|\, \mathrm{ad}(X) \text{ is nilpotent}\}.$$

From these remarks, using the finiteness theorem for nilpotent orbits [17,§3], we have that $\mathcal{V}(\mathrm{gr}I_V)$ is a finite union of nilpotent orbits. But, even more is true [6]:

$$\mathcal{V}(\mathrm{gr}I_V) = \overline{\mathcal{O}_V}$$

for some nilpotent orbit $\mathcal{O}_V$. These remarks describe the desired assignment

$$V \rightsquigarrow \mathcal{O}_V. \tag{1.1a}$$

We sometimes refer to $\mathcal{O}_V$ as the *nilpotent orbit associated to* $V$. Define the Gelfand-Kirillov dimension of $V$ to be $\mathrm{Dim} V = \frac{1}{2}\dim_{\mathbb{C}}\mathcal{O}_V$; every coadjoint orbit carries a symplectic structure, which insures its dimension is even [17,§1.4].

For our needs, one type of nilpotent orbit is of particular interest. The *Richardson Orbit* $\mathcal{O}_{\mathfrak{p}}$ associated to the parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the unique nilpotent orbit in $\mathfrak{g}$ which is dense in $\mathrm{Ad}(G_{ad}).\mathfrak{n}$; this orbit is denoted $\mathcal{O}_{\mathfrak{p}}$. For more details, see [17,§7].

Given a Harish-Chandra module $V$ in $\mathcal{HC}_o$, define $J(V) = (\tilde{V})^*_{\mathfrak{b}-\text{locally finite}}$, where $\tilde{}$ (resp. $*$) refers to the admissible (resp. full) dual of $V$. This assignment defines a faithful exact covariant functor. We refer to $J(V)$ as the *Jacquet module of* $V$. The module $J(V)$ lies in the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$, for all $V \in \mathcal{HC}_o$.

To make sense of one of our introductory remarks, it is important to recall

$$H^k(\mathfrak{n}_m, J(\bar{\pi}(\delta))) = H_k(\bar{\mathfrak{n}}_m, \bar{\pi}(\delta)), \tag{1.1b}$$

for all $k \in \mathbb{N}$. Information about submodules of $J(\bar{\pi}(\delta))$ will be encoded by highest weight vectors contributing to $H^o(\mathfrak{n}_m, J(\bar{\pi}(\delta)))$; combined with Frobenius reciprocity [20] we obtain embeddings of $\bar{\pi}(\delta)$ into principal series representations.

We seek to link the existence of "nice submodules" of $J(\bar{\pi}(\delta))$ with a condition on the nilpotent orbit $\mathcal{O}_{\bar{\pi}(\delta)}$. To carefully define these "nice submodules", define

$$W^P_{\text{soc}} = \{\, w \mid w \in W^P, L_{\mathfrak{p}}(w) \subset \text{socle}(N_{\mathfrak{p}}(y)), \text{ for some } y \in W^P \,\}, \tag{1.1c}$$

which is referred to as the *socular set for* $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$. This set is parametrized by those $L_{\mathfrak{p}}(w)$ with the property that $\mathrm{Dim} L_{\mathfrak{p}}(w) = \dim \mathfrak{n}$. For example, if $\mathfrak{p} = \mathfrak{b}$, then $W^P_{\text{soc}} = \{e\}$. Roughly speaking, as $\mathfrak{p}$ gets "bigger", the size of the socular set increases and the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ gets "smaller". The importance of this set is clearly spelled out in Irving's work [21]. We now come to a central definition.

**Definition 1.2.** *Let $\bar{\pi}(\delta)$ be an irreducible Harish-Chandra module for $G$ and $\mathfrak{p}$ a standard parabolic subalgebra of $\mathfrak{g}$. We say that $\bar{\pi}(\delta)$ has Property $\mathfrak{p}$ if there exists an irreducible highest weight module $L$ satisfying two conditions:*

*(a) $L$ lies in the socle of $J(\bar{\pi}(\delta))$;*

*(b) $L = L_{\mathfrak{p}}(w)$ for some $w \in W^P_{\text{soc}}$.*

*Any such $L$ satisfying (a) and (b) is called a $\mathfrak{p}$-factor for $J(\bar{\pi}(\delta))$.*

Given $\mathfrak{p}$, an obvious problem is to classify the irreducible Harish-Chandra modules having Property $\mathfrak{p}$. It is fairly easy to give a necessary condition; see §2 for a proof.

**Lemma 1.3.** *If $\bar{\pi}(\delta)$ has Property $\mathfrak{p}$, then $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathfrak{p}}$.*

Any hope of establishing the converse of (1.3) requires a more careful hypothesis on $\mathfrak{p}$. (As will become clear in the sequel, without additional hypothesis the converse of (1.3) fails.) A *Whittaker datum* $\Psi$ is a triple $(P, \psi, \mathfrak{n})$, where $P = MAN$ is the Langlands decomposition of a parabolic subgroup of $G$ and $\psi$ is a character (one-dimensional representation) of $\mathfrak{n}$. We say that the Whittaker datum $\Psi$ is *admissible* if the Richardson orbit associated to $\mathfrak{p}$ coincides with the orbit determined by $\psi$; i.e. $\mathcal{O}_{\mathfrak{p}} = G_{ad}.\psi$. It is not true that all parabolic subalgebras admit admissible Whittaker datum. However, in §2 we establish the following well-known result; it insures the main results of this paper are not vacuous.

**Lemma 1.4.** *Let $\mathfrak{p}$ be an even Jacobson-Morozov parabolic subalgebra of $\mathfrak{g}$ arising as the complexification of a real parabolic subalgebra of $\mathfrak{g}_o$. Then $\mathfrak{p}$ admits admissible Whittaker datum.*

If $\mathcal{A}(G)$ is the space of real analytic functions on $G$, then under the left action we have the induced representation $\mathcal{A}(G;\Psi)$, which is just the space of real analytic sections of the line bundle over $G/N$ determined by the one-dimensional representation $e^{-\psi}$. Given an arbitrary $\mathfrak{U}(\mathfrak{g})$-module $V$, if there exists an injective $\mathfrak{U}(\mathfrak{g})$-homomorphism $i: V \mapsto \mathcal{A}(G;\Psi)$, then we will say $V$ has a *$\Psi$-global Whittaker model.*

Using our terminology, the next result was established by Matumoto, generalizing earlier work of Goodman-Wallach.

**Theorem 1.5 (Goodman-Wallach [19], Matumoto [26]).** *Fix $\Psi$ an admissible Whittaker datum for $G$ and $\bar{\pi}(\delta)$ an irreducible Harish-Chandra module. If $\bar{\pi}(\delta)$ has Property $\mathfrak{p}$, then $\bar{\pi}(\delta)$ has a $\Psi$-global Whittaker model.*

This leads us to our main problem of interest: Give a necessary and sufficient condition for the existence of a $\Psi$-global Whittaker model for $\bar{\pi}(\delta)$; or equivalently, necessary and sufficient conditions for Property $\mathfrak{p}$.

**Matumoto's Conjecture 1.6.** *Let $\mathfrak{p}$ be an even Jacobson-Morozov parabolic subalgebra defined over $\mathbb{R}$ and $\mathcal{O}_{\mathfrak{p}}$ the corresponding Richardson orbit. Fix $\Psi$ an admissible Whittaker datum for $G$ and assume that $\bar{\pi}(\delta)$ is an irreducible Harish-Chandra module with $Dim\bar{\pi}(\delta) = dim\mathfrak{n}$. The following are equivalent:*

*(a) (Singularity Condition) $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathfrak{p}}$;*
*(b) (Whittaker Condition) $\bar{\pi}(\delta)$ has a $\Psi$-global Whittaker model;*
*(c) (Asymptotic Condition) $\bar{\pi}(\delta)$ has Property $\mathfrak{p}$.*

Matumoto has made significant progress on this conjecture. First, in [25] he showed that (b) implies (a) and (1.5) is (c) implies (b). In case $P = P_m$, Casselman's Subrepresentation Theorem shows that the Singularity Condition implies the Asymptotic Condition (and hence, the Whittaker Condition). In addition, when $G$ is a complex group, Matumoto [27] additionally established (a) implies (c), whence proving the conjecture. The implication "$(a) \Longrightarrow (c)$" is sometimes referred to as "The working hypothesis". We can now state the first main result of this paper.

**Theorem 1.7.** *If $G$ is of Hermitian symmetric type and of real rank two, then Matumoto's conjecture is true.*

In §9 we will give a detailed account of the validity of the working hypothesis in the case of $Sp_6\mathbb{R}$ and offer counterexamples to (1.6).

**Proposition 1.8.** *In the case of $Sp_6\mathbb{R}$, the fundamental block of the finite dimensional representation $F$ is a union of 16 double cells. Matumoto's conjecture is true on all but two of these double cells. On these two double cells the conjecture fails (i.e. the working hypothesis (a) implies (c) in (1.6) fails).*

In this sense, without further restricting the groups in question or representations of interest, (1.7) is the best general statement one can make. (We should point out that H. Matumoto has informed the author of counterexamples in $Sp_6\mathbb{R}$ using very different techniques.)

One might naturally ask to what extent one can remove the Hermitian symmetric hypothesis in (1.7). To comment on this, lets first recall the list of simple real rank two matrix groups, up to covering, amounts to 4 infinite families and 7 sporadic cases:

| *Hermitian Symmetric* | *Non-Hermitian Symmetric* |
|---|---|
| $SU(2,q)$ | $Sp(2,s)$ |
| $SO_e(2,2n-1)$ | |
| $SO_e(2,2n-2)$ | |
| | |
| $SO^*(10)$ | $Sl_3\mathbb{R}$ |
| $Sp_4\mathbb{R}$ | $Sl_3\mathbb{H}$ |
| $E_{6(-14)}$ | $E_{6(-26)}$ |
| | $G_{2(2)}$ |

In §10, we address the non-Hermitian cases. We will see, in the case of $Sl_3\mathbb{R}$, $Sl_3\mathbb{H}$ and $E_{6(-26)}$, the only even Jacobson-Morozov parabolic defined over $\mathbb{R}$ is the minimal parabolic $\mathfrak{p}_m$ and in this setting (1.6) follows from Matumoto's work in [26]. The case of $G_{2(2)}$ is non-trivial, but still we are able to prove (1.6). This leaves the infinite family $Sp(2,s)$. We have verified (1.6) in the case of $s=2$, but a general proof would require tools in the spirit of [5], which are currently unavailable. The ideas and techniques of proof we use for (1.7) will build upon the material in the Memoir [5], which was cast entirely in the Hermitian symmetric setting. Never the less, if (1.6) holds for the cases $s \geq 3$, we would then be able to remove the "Hermitian symmetric" assumption in (1.7).

Here is a brief outline of the content of each section of the paper. In §2, we introduce the necessary notation and terminology, most of which is standard. Section 3 will establish a useful reduction lemma; in effect, we are reduced to verifying (1.7) for one irreducible representation from each relevant double cell. This result is really a manifestation of the fact that the Jacquet functor "intertwines" double cell and right cell Weyl group representations. Section 4 outlines the basic strategy used in our proof of (1.7). The proof of the main result (1.7) is carried out in §5-8 and $Sp_6\mathbb{R}$ is studied in §9. Non-Hermitian real rank two groups are discussed in §10.

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