Stretching Pythagoras Around the Corner: Linking and Modeling in Precalculus

by David H. Collingwood and Marilyn Stor [1]

TEACHER'S GUIDE

Students engage in an experiential and theoretical application of a problem involving a *bungee cord.* A hands-on modeling approach is described, using elastic, pencils and graph paper. The activity motivates geometry, coordinate systems, rates, linear applications and the concept of a function. Possible extensions focus on vectors, parametric equations and trigonometry. The activity resurfaces several times in a problem solving precalculus course[2] [1].

The situation underlying the activity is summarized in Figure 1: Allyson and Adrienne have connected their ankles with a *bungee cord.*

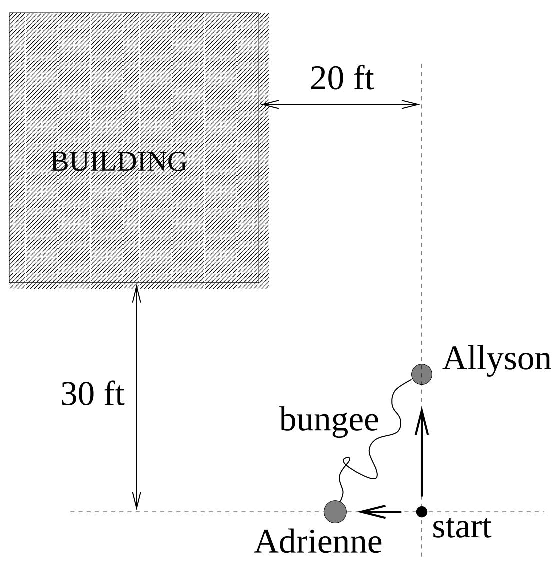

Fig. 1: The problem

They start at the same location near the outside corner of a building (as pictured from above) and move in the directions indicated. The activity investigates a number of questions involving

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[2] This problem was presented by the authors at the NCTM annual meeting, April 24, 1999.

the length of the bungee cord, the locations of the two girls and the relationship between the bungee cord and the building.

You may wish to show the students a sample of *bungee cord*, which can be purchased at your local hardware store. This activity has been carried out using a "full-scale model" with students working in teams outdoors.

The three activity sheets each depend on their predecessors. The objectives and answers for each activity sheet are described below.

Getting Started

Students should work in pairs. Each pair will need: Three activity sheets, two pencils, 12 inches of elastic cord (the type used in sewing), one photocopy of the "master worksheet" magnified to 125% (or larger) onto 11×17 paper (the bigger the better).

The first step involves tying the elastic cord between two pencils so as to accurately model a bungee cord which stretches to a maximum length of 90 feet. To do this, students use the "master worksheet" as follows: The grids of the "master worksheet" will be taken to model 10 ft. increments. The student can use this grid to accurately tie the elastic between the two pencils so that when <u>fully stretched</u> the elastic spans 9 grid units. Next, they determine the length of the modeled un-stretched bungee cord and record this at the top of the "master worksheet" as "ℓ feet". A typical value for $\ell = 30$ ft., but this will vary depending on the "stretchiness" of the elastic cord used.

Because elastic cord can deform after repeated use, it is a good idea to repeat the above calibration process at the beginning of each activity sheet.

REFERENCES

[1] Precalculus: A first course in problem solving, David Collingwood, University of Washington class notes, 500 pages, 1995-01 editions, Professional Copy and Print, Seattle, WA. This text is freely available on line (login="precalculus"; password="UW120rocks") at

`http://www.math.washington.edu/~colling/HSMath120/home.html`

Begin the activity by specifying rates: Allyson moves 10 ft/sec and Adrienne 8 ft/sec in the directions indicated in Figure 1. Students record these rates at the top of the "master worksheet". The model begins with each student holding a pencil at the starting location. One student will act as the "clock" and count off "one-thousand one, one-thousand two, one-thousand three..." etc. When a time is called out, the pencil representing Adrienne will move 8 ft. and the pencil representing Allyson will move 10 feet. Since our paper grid is in 10 ft. units, we have marked off even foot subdivisions along the horizontal line representing Adrienne's motion. See Figure 2.

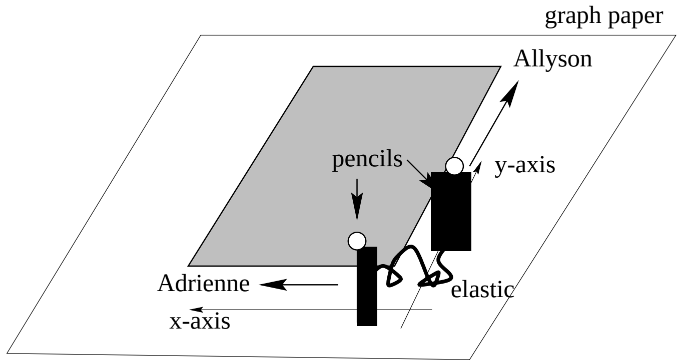


Fig. 2: A 3D picture of the model

Objectives: Students determine when the bungee cord first touches the corner of the building. They work with the hands-on model to estimate answers, think about a coordinate system and work with the geometry of right triangles. A multipart function modeling the length of the bungee cord is derived. The questions should be worked in the order listed.

Answers: 1. Students can give precise answers using the imposed coordinate system.

time (seconds)	Allyson's coordinates	Adrienne's coordinates
1	$(0, 10)$	$(-8, 0)$
2	$(0, 20)$	$(-16, 0)$
4	$(0, 40)$	$(-32, 0)$

2. Students discover the slack is taken up before the bungee cord reaches the building corner. Determining precisely when the slack is taken up will involve a right triangle; you may wish to give the hint: "think right triangle". If ℓ feet is the un-stretched bungee cord length, then study the right triangle with hypotenuse ℓ, vertical side $10t$ and horizontal side $8t$. Solving this right triangle, we find $t = \sqrt{\frac{\ell^2}{164}}$; so, Allyson's location $= (0, 10\sqrt{\frac{\ell^2}{164}})$ and Adrienne's location $= (-8\sqrt{\frac{\ell^2}{164}}, 0)$. (For example, if $\ell = 30$ ft, then $t = 2.3426$ sec., etc.)

3. They can get a rough time estimate using the hands-on model; i.e. between 5 and 6 seconds. You may wish to give the hint: "think similar triangles". The key is to obtain a picture as in Figure 3.

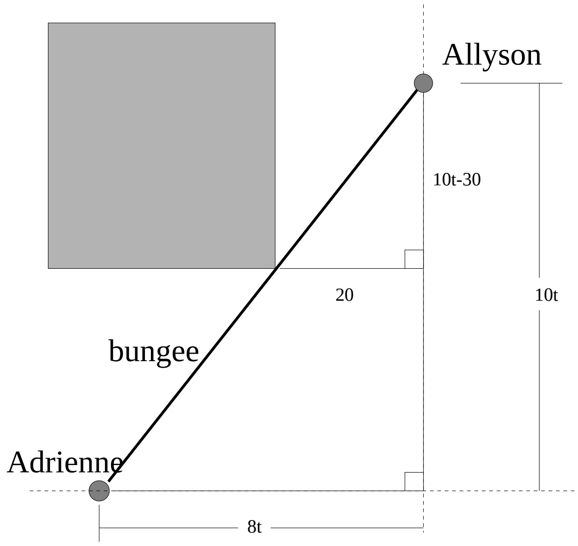


Fig. 3: The bungee touches the building

Equating the ratios of labeled sides of similar triangles:

$$\frac{10t}{8t} = \frac{10t - 30}{20}; \quad t = 5.5 sec.$$

4. The function $b(t)$ is multipart, depending on two cases: (i) times when the bungee cord is slack; (ii) times when the bungee cord is straight and partially stretched. The domain of the function is $0 \leq t \leq 5.5$, by question 3. In question 2., $t = \sqrt{\frac{\ell^2}{164}}$ sec. is the time when the bungee cord first becomes tight. Thus, the multipart rule will be

$$b(t) = \begin{cases} \ell, & \text{if } 0 \leq t \leq \sqrt{\frac{\ell^2}{164}} \\ \sqrt{(8t)^2 + (10t)^2}, & \text{if } \sqrt{\frac{\ell^2}{164}} \leq t \leq 5.5 \end{cases}$$

Guide to Sheet #2

This activity will determine when the bungee cord becomes fully stretched and where the girls are located at this instant. It is important to emphasize the assumptions concerning the motion of the two girls for this activity sheet:

- Both girls begin at the starting location (as in Sheet #1), with Allyson moving 10 ft/sec in the positive y-direction and Adrienne moving 8 ft/sec in the negative x-direction.
- Adrienne stops moving at time $t = 5.5$ seconds while Allyson continues to move until the bungee cord is fully stretched.

Students should continue to use their "master worksheet". During this activity, one of the students in each team will need to place a finger from their free hand at the building corner on the "master worksheet". This allows the bungee cord to "bend around the corner" when the time passes $t = 5.5$ seconds.

Objectives: Students use the Pythagorean Theorem twice on two non-similar right triangles. A multipart function that models the length of the bungee cord must be derived.

Answers: 1. From activity sheet #1, the partially stretched bungee first touches the building at time $t = 5.5$ seconds. Students should draw pictures as in Figure 4.

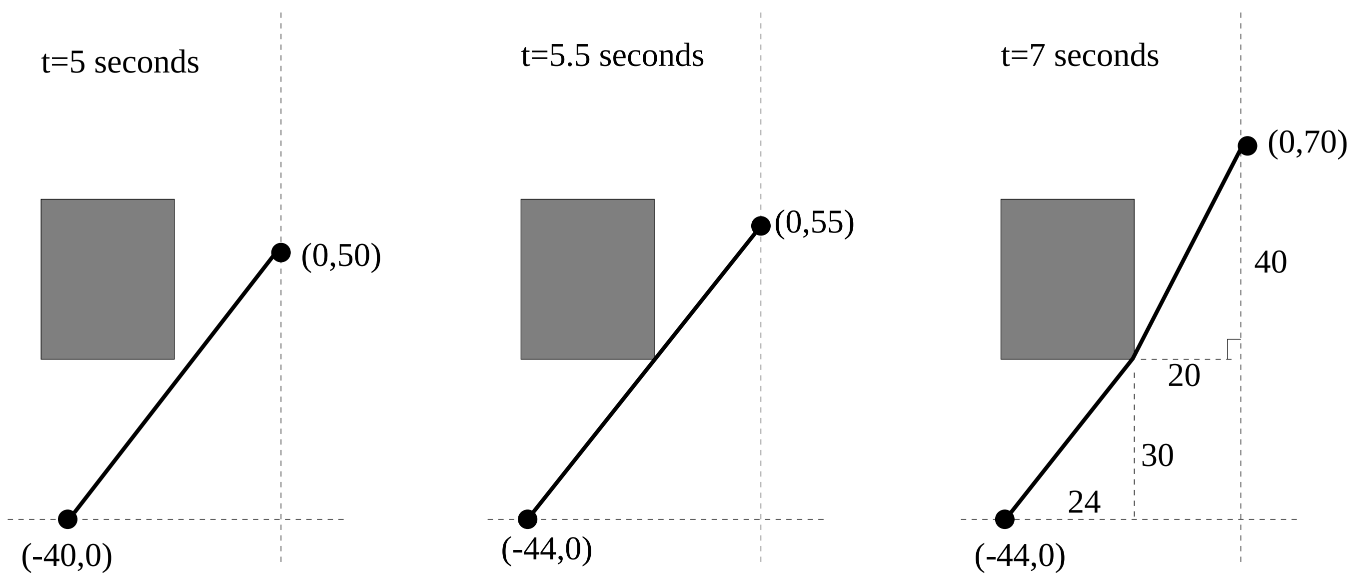


Fig. 4: Bending around the corner

Pay special attention that Adrienne's location is the same at times $t = 5.5$ and 7 seconds, due to our assumption on Adrienne's motion. The length of the bungee cord at times $t = 5$ and 5.5

seconds are easily given by the Pythagorean Theorem. In the case of $t = 7$ seconds, use the Pythagorean Theorem twice, as suggested by the two inscribed right triangles in Figure 4.

time (seconds)	Length of stretched bungee cord
5	$\sqrt{40^2 + 50^2} = 64.03$ feet
5.5	$\sqrt{44^2 + 55^2} = 70.43$ feet
7	$\sqrt{24^2 + 30^2} + \sqrt{20^2 + 40^2} = 83.14$ feet

2. Use the hands-on model to estimate the bungee is fully stretched between 7 and 8 seconds.

3. To solve the problem, a picture as in Figure 5 will arise involving two non-similar right triangles.

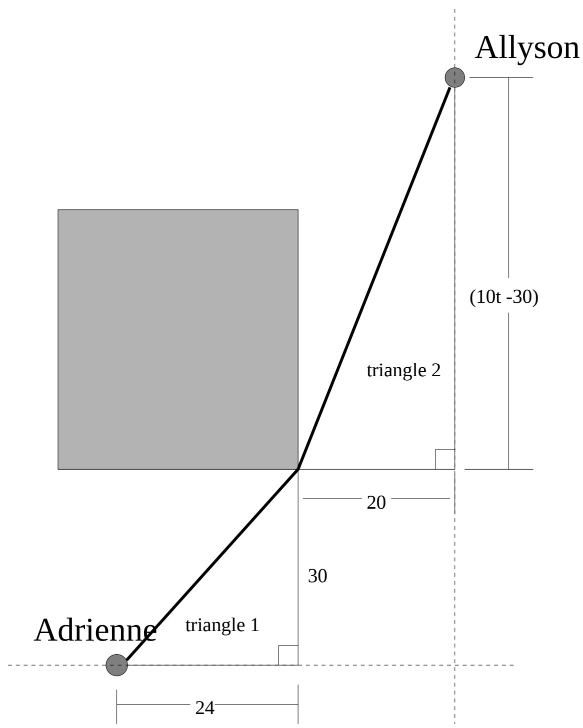


Fig. 5: Fully stretched bungee scenario

The sum of the two triangle hypotenuses should be 90 feet, giving us the equation:

$$90 = \sqrt{24^2 + 30^2} + \sqrt{20^2 + (10t - 30)^2}.$$

This can be solved algebraically or with an equation solver; the solution is $t = 7.7546$ seconds. (Note: I like to solve this without using an equation solver, because it will feed into a common error students make in the third activity sheet.) The EXACT solution is $t = \frac{1}{10}(30 + \sqrt{(90 - \sqrt{1476})^2 - 400})$. Allyson is located at $(0, 77.55)$.

4. The formula for $b(t)$ is multipart, depending on these three cases: (i) times when the bungee cord is slack; (ii) times when the bungee cord is partially stretched AND straight; (iii) times when

the bungee cord is partially stretched AND bent. Activity sheet #1 gives us the formula for $b(t)$ on the domain $0 \leq t \leq 5.5$. The previous question shows the domain of $b(t)$ is $0 \leq t \leq 7.7546$ seconds. For times $5.5 \leq t \leq 7.7546$ seconds, use Figure 5 to give the third part of the multipart rule:

$$b(t) = \begin{cases} \ell, & \text{if } 0 \leq t \leq \sqrt{\frac{\ell^2}{164}} \\ \sqrt{(8t)^2 + (10t)^2}, & \text{if } \sqrt{\frac{\ell^2}{164}} \leq t \leq 5.5 \\ \sqrt{24^2 + 30^2} + \sqrt{20^2 + (10t - 30)^2}, & \text{if } 5.5 \leq t \leq 7.7546 \end{cases}$$

Guide to Sheet #3

This activity will determine when the bungee cord becomes fully stretched and where the girls are located the instant this occurs. Here are the assumptions we make for this activity.

- Both girls begin at the starting location on the "master worksheet", with Allyson moving 10 ft/sec in the positive vertical direction and Adrienne moving 8 ft/sec in the negative horizontal direction.

- Both girls continue walking until the bungee cord is fully stretched.

Students should continue to use their "master worksheet". In this activity, the time when the bungee cord is fully stretched cannot be solved algebraically and you will need a graphing calculator or similar technology. As before, one of the students in each team will need to place a finger from their free hand at the building corner on the "master worksheet". This allows the bungee cord to "bend around the corner" when the time passes $t = 5.5$ seconds.

Objectives: This activity again requires the students use the Pythagorean Theorem. A radical equation must be solved which can feed into a common student algebra error; the correct procedure requires technology. A multipart function that models the length of the bungee cord at time t seconds must be derived. This builds on the previous activity.

Answers: 1. Again, from activity sheet #1, the partially stretched bungee cord first touches the building at time $t = 5.5$ seconds. The student should draw in pictures as in Figure 6.

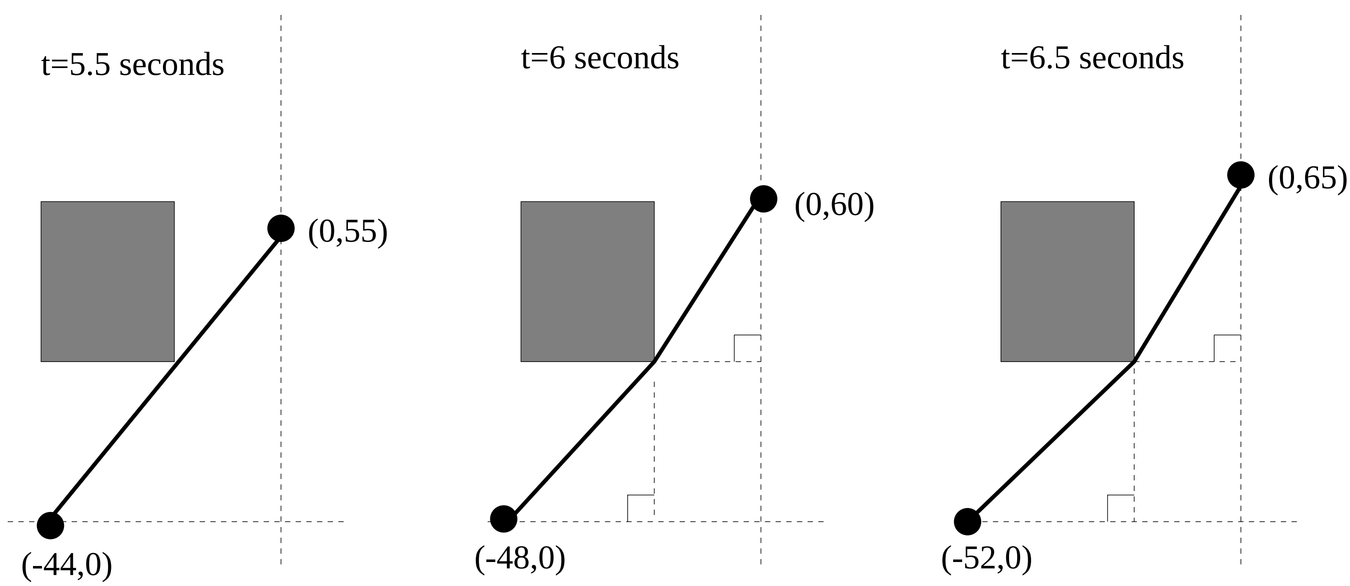


Fig. 6: Bending around the corner

time (seconds)	Length of stretched bungee cord
5.5	$\sqrt{44^2 + 55^2} = 70.43$ feet
6	$\sqrt{(48-20)^2 + 30^2} + \sqrt{(60-30)^2 + 20^2} = 77.09$ feet
6.5	$\sqrt{(52-20)^2 + 30^2} + \sqrt{(65-30)^2 + 20^2} = 84.17$ feet

2. Use the hands-on model to estimate the bungee cord is fully stretched between 6 and 8 seconds.

3. To solve the problem, a picture as in Figure 7 will arise.

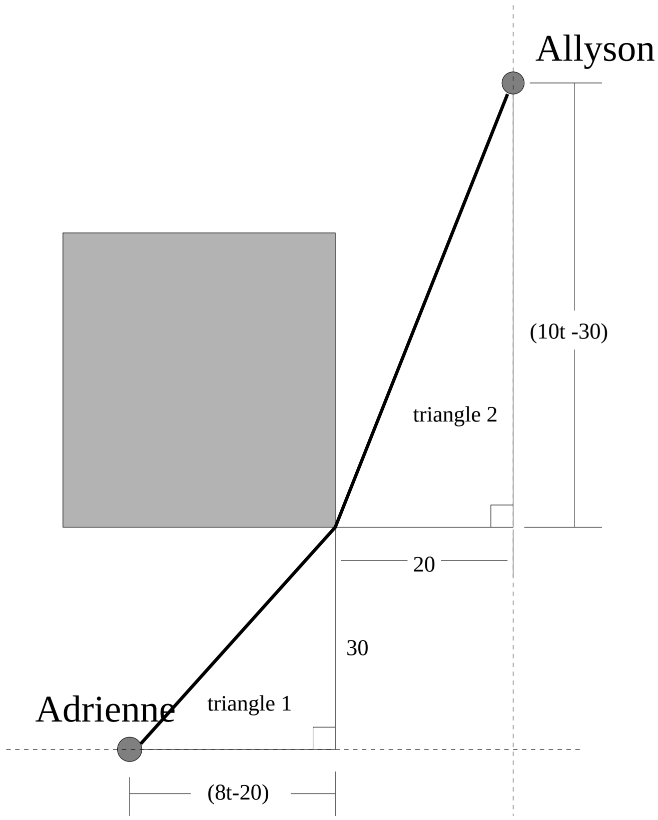


Fig. 7: Fully stretched bungee scenario

Working with two non-similar right triangles leads to the equation

$$90 = \sqrt{(8t-20)^2 + 30^2} + \sqrt{20^2 + (10t-30)^2}.$$

If you try to proceed algebraically, as in activity sheet #2, some students will make a deadly error: they will square both sides and think that this simply eliminates the square roots. (This is an error I continue to see in my university courses and is worth some attention.) Of course, this reasoning is faulty and there is no algebraic way to solve the equation. Instead, students will need to use zooming or equation solver techniques and arrive at $t = 6.895$ seconds as the time when the bungee cord is fully stretched; Allyson is located at $(0, 68.95)$.

4. The function $B(t)$ is multipart, depending on these three cases: (i) times when the bungee cord is slack; (ii) times when the bungee cord is partially stretched AND straight; (iii) times when the bungee cord is partially stretched AND bent. The first activity sheet gives the formula for $B(t)$ on the domain $0 \le t \le 5.5$. The previous question shows the domain of the function is

$0 \le t \le 6.895$ seconds. For times $5.5 \le t \le 6.895$ seconds, we use Figure 7 to obtain the third part of the multipart rule:

$$B(t) = \begin{cases} \ell, & \text{if } 0 \le t \le \sqrt{\frac{\ell^2}{164}} \\ \sqrt{(8t)^2 + (10t)^2}, & \text{if } \sqrt{\frac{\ell^2}{164}} \le t \le 5.5 \\ \sqrt{(8t-20)^2 + 30^2} + \sqrt{20^2 + (10t-30)^2}, & \text{if } 5.5 \le t \le 6.895 \end{cases}$$

Variation #1: This variation involves perpendicular lines, intersections of lines, the distance formula and parametric equations.

Assume the scenario of activity sheet #1. Find a function that computes the shortest distance from the corner of the building to the partially stretched bungee cord at time t.

To solve this, construct a line passing through the building corner and perpendicular to the partially stretched bungee cord; see Figure 8.

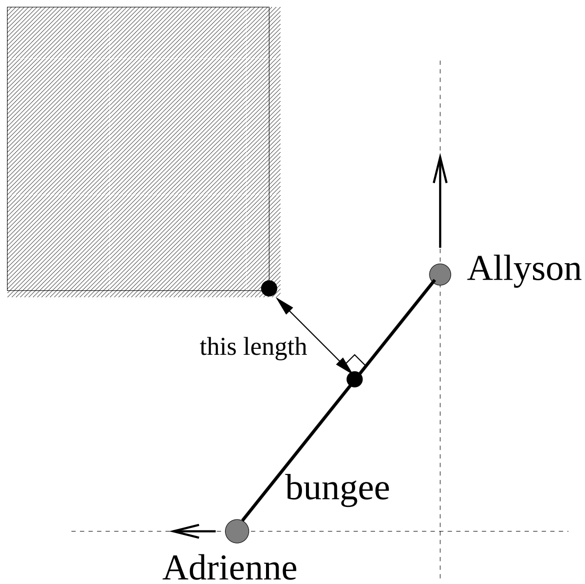


Fig. 8: Distance problem

The distance between the corner and the intersection point of these two lines is the key quantity to calculate. An equation for the line modeling the partially stretched straight bungee cord at time t is $y = \frac{5}{4}x + 10t$; discuss with the students why this model is only valid on the domain $\sqrt{\frac{\ell^2}{164}} \leq t \leq 5.5$. The perpendicular line through the building corner has equation $y = -\frac{4}{5}(x + 20) + 30$. Simultaneously solve these two linear equations for x and y in terms of t to get the closest point being $P = (x(t), y(t))$, where $x(t) = \frac{-200t}{41} + \frac{280}{41}$, $y(t) = \frac{160t}{41} + \frac{350}{41}$. Use the distance formula between $(-20, 30)$ and P to get the distance is

$$d(t) = \sqrt{(\frac{-200t}{41} + \frac{1100}{41})^2 + (\frac{160t}{41} - \frac{880}{41})^2}.$$

Variation #2: This variation works with angles and inverse trig functions. Students construct a particular function whose basic properties can be predicted using the hands on model.

Continue to assume the scenario on activity sheet #3. After the bungee first touches the building, it will form an angle of measure greater than 180 degrees; see Figure 9.

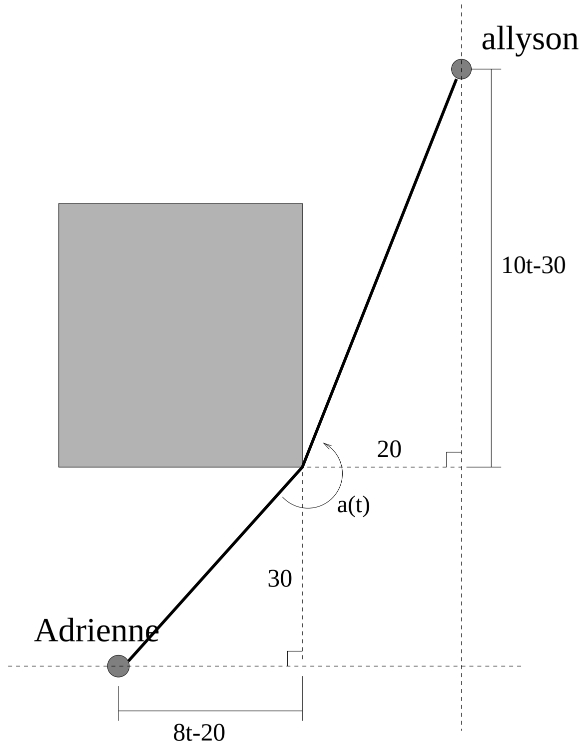


Fig. 9: Determining the bent angle of the bungee cord

Find a function $a(t)$ that computes this angle at time t seconds. Does the angle ever exceed 220^o? What is the largest this angle can be?

This problem will require the use of some inverse trigonometric functions and a discussion of an appropriate function domain to get

$$a(t) = \frac{\pi}{2} + \arctan\left(\frac{10t-30}{20}\right) + \arctan\left(\frac{8t-20}{30}\right).$$

Students can study the graph of this function, see it is increasing on the domain $5.5 \leq t \leq 6.895$ and deduce that the maximum angle measure is 211.6^o, using technology. This is a good one for the students to estimate hands-on, using the model and a protractor.

Variation #3: This variation involves either changing the rate the girls are moving, changing the starting locations of each girl (which need not coincide as in the original problem) or changing the length of the bungee cord. There are countless variations on this theme.

Variation #4: This variation involves changing the directions the girls move and naturally leads to a discussion of vectors and parametric equations.

For example, we might assume that Allyson moves 10 ft/sec and Adrienne moves 8 ft/sec in the directions pictured in Figure 10.

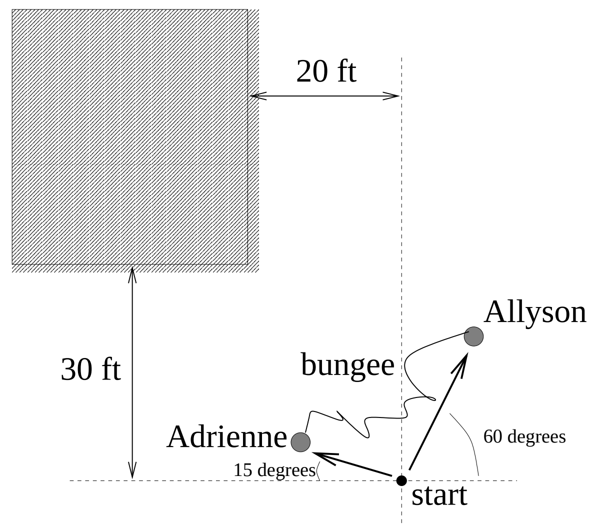
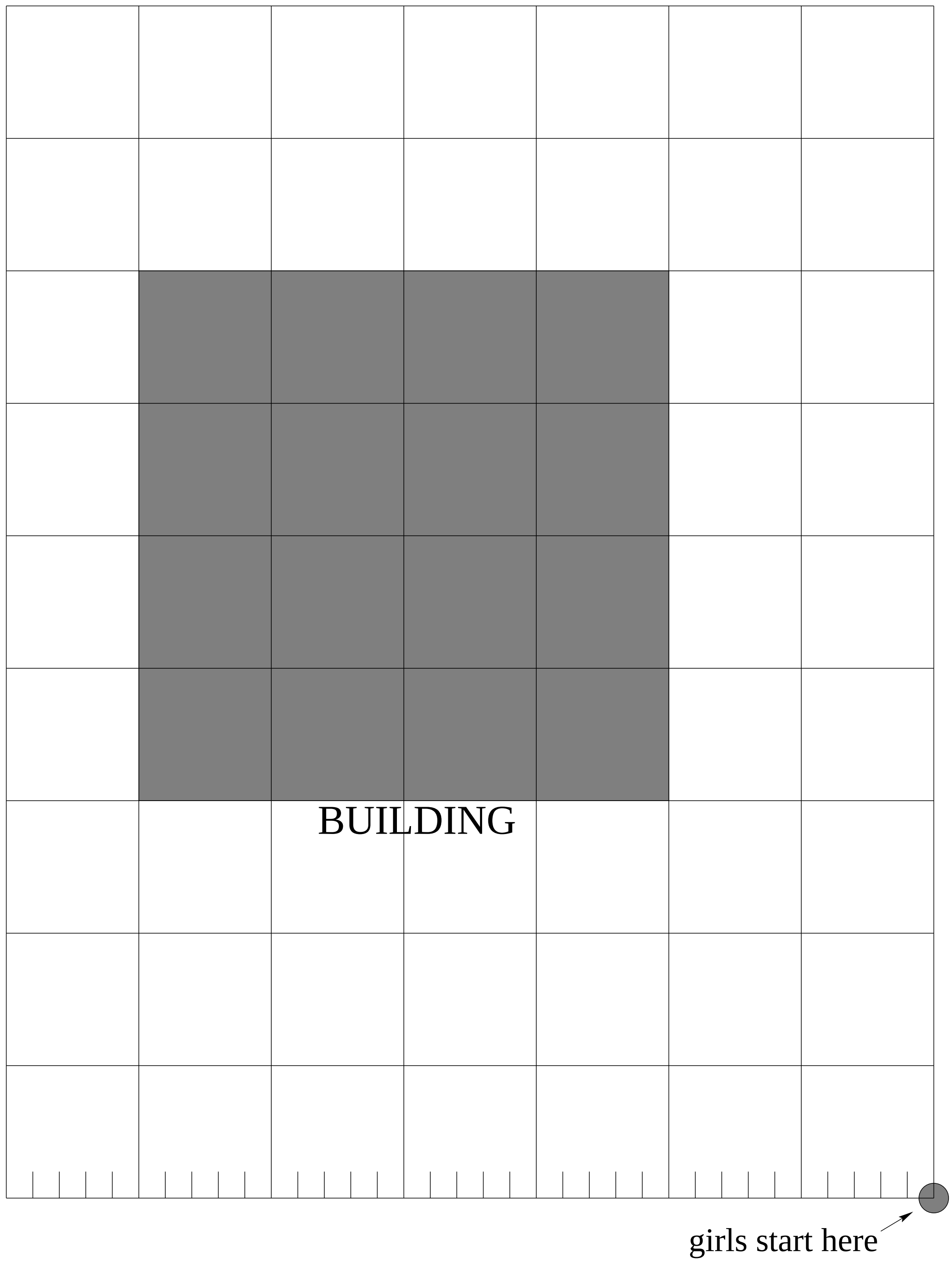


Fig. 10: Bungee problem vector variation

One can resolve the motion into x and y components and find parametric equations for the linear motion of each girl. Then ask questions as before. Depending on the setup, one interesting new type of question arises: Does Adrienne or the bungee cord hit the building before the bungee cord is fully stretched?

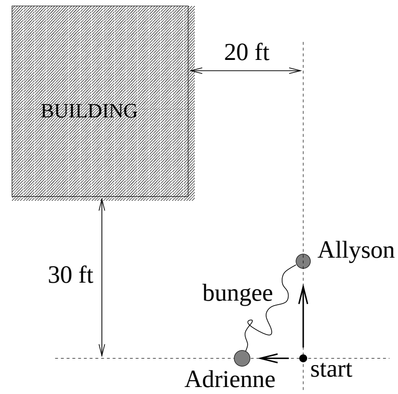
MASTER WORKSHEET

Allyson and Adrienne start moving from the same location relative to the building corner. The grid units are 10 feet. Allyson moves ________ feet/sec in the positive vertical direction and Adrienne moves ________ feet/sec in the negative horizontal direction. The fully stretched bungee cord has length 90 feet. The un-stretched bungee cord has length $\ell =$ ________ feet.



SHEET #1

Allyson and Adrienne have connected their ankles with a *bungee cord*. Assume the fully stretched bungee cord has length 90 feet; the un-stretched bungee cord has length ℓ feet, as computed at the top of your "master worksheet". Both girls start at the same location near the outside corner of a building. Allyson moves 10 ft/sec and Adrienne moves 8 ft/sec, along the indicated lines of motion.



First Question Set: On your "master worksheet", impose (draw in) a coordinate system with the starting location as the origin; label the x and y axes.

1. Mark the locations of the two girls after 1 seconds, 2 seconds and 4 seconds. Tabulate their coordinates.

time (sec)	Allyson's coords	Adrienne's coords
1		
2		
4		

2. Determine when the bungee cord first becomes tight. (This is when all slack has been removed AND the bungee cord is not yet stretching.) Where are the girls located when this occurs?

3. Use the hands-on model to estimate when the bungee cord touches the building corner, then determine this time precisely.

4. Find the formula for a function $b(t)$ that gives the length of the bungee cord up to the instant the bungee cord touches the building corner.

SHEET #2

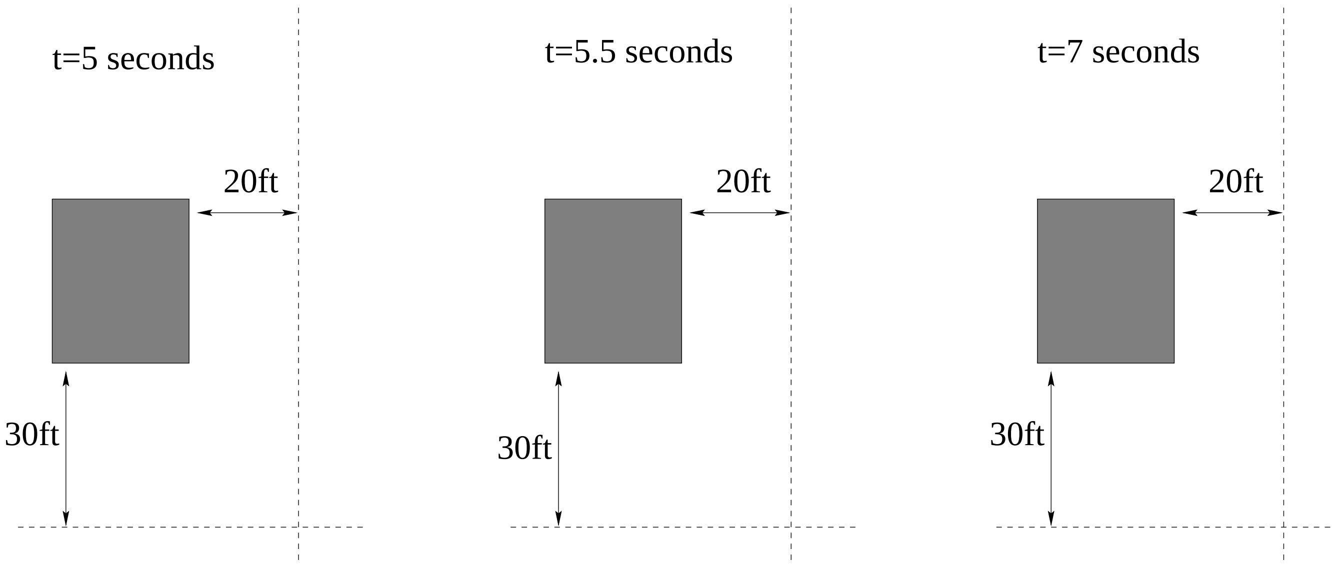
The goal is to determine when the bungee cord becomes fully stretched and where the girls are located the instant this occurs. Here are the assumptions we make for this activity.

- Both girls begin at the starting location on the "master worksheet", with Allyson moving 10 ft/sec in the positive vertical direction and Adrienne moving 8 ft/sec in the negative horizontal direction.
- Adrienne stops moving at time $t = 5.5$ seconds while Allyson continues to move until the bungee cord is fully stretched.

Second Question Set:

1. Sketch the bungee cord and locations of Allyson and Adrienne at times $t = 5, 5.5$ and 7 seconds in the figure provided. Fill out the table, calculating the length of the stretched bungee cord at each time.

time (seconds)	Length of stretched bungee cord
5	
5.5	
7	



2. Use the hands-on model to estimate when the bungee cord becomes fully stretched.
3. Where is Allyson when the bungee cord reaches its maximum length?
4. Find the formula for the function $b(t)$ that computes the length of the bungee cord at time t seconds.

SHEET #3

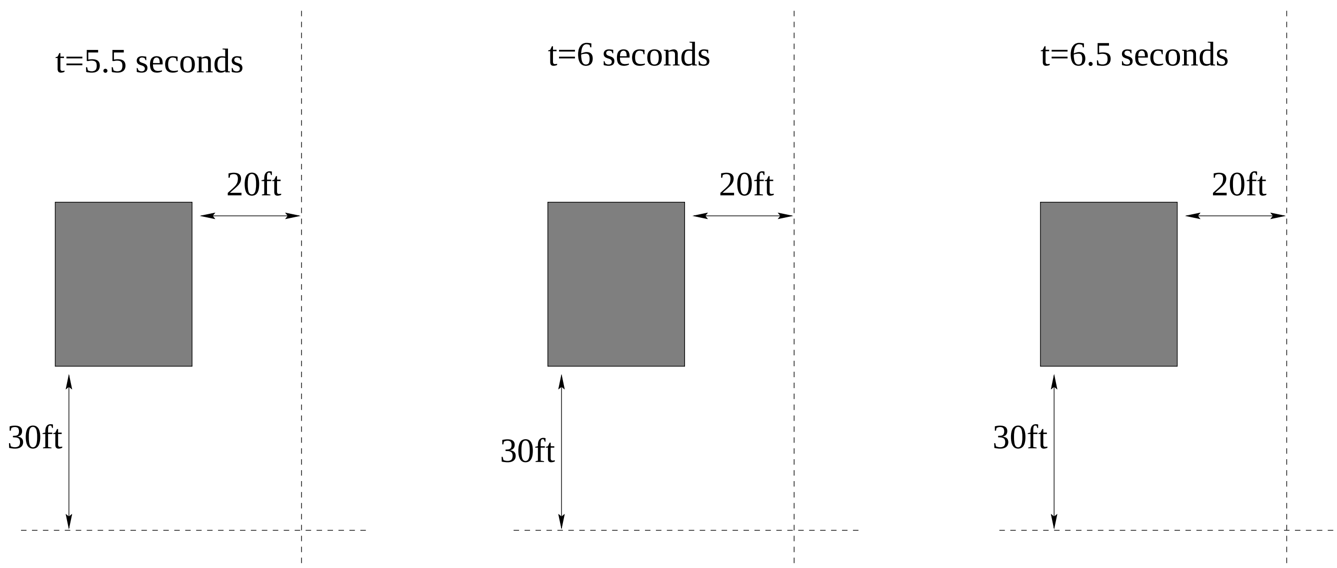
The goal is to determine when the bungee cord becomes fully stretched and where the girls are located the instant this occurs. Here are the assumptions we make for this activity.

- Both girls begin at the starting location on the "master worksheet", with Allyson moving 10 ft/sec in the positive vertical direction and Adrienne moving 8 ft/sec in the negative horizontal direction.
- Both girls continue walking until the bungee cord is fully stretched.

Third Question Set:

1. Sketch the bungee cord and locations of Allyson and Adrienne at times $t = 5.5, 6$ and 6.5 seconds in the figure provided. Fill out the table, calculating the length of the bungee cord at each time.

time (seconds)	Length of stretched bungee cord
5.5	
6	
6.5	



2. Use the hands-on model to estimate when the bungee becomes fully stretched.

3. Where is Allyson when the bungee reaches its maximum length?

4. Find a formula for the function $B(t)$ that computes the length of the bungee cord at time t seconds.