DETERMINANTS AND PATH COUNTING

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Main idea:

- Start with a (weighted) network.
- Write down a matrix that encodes "how much gets from each source to each sink" in the network.
- Relate subdeterminants to path "counting".

Objects we need to know about:

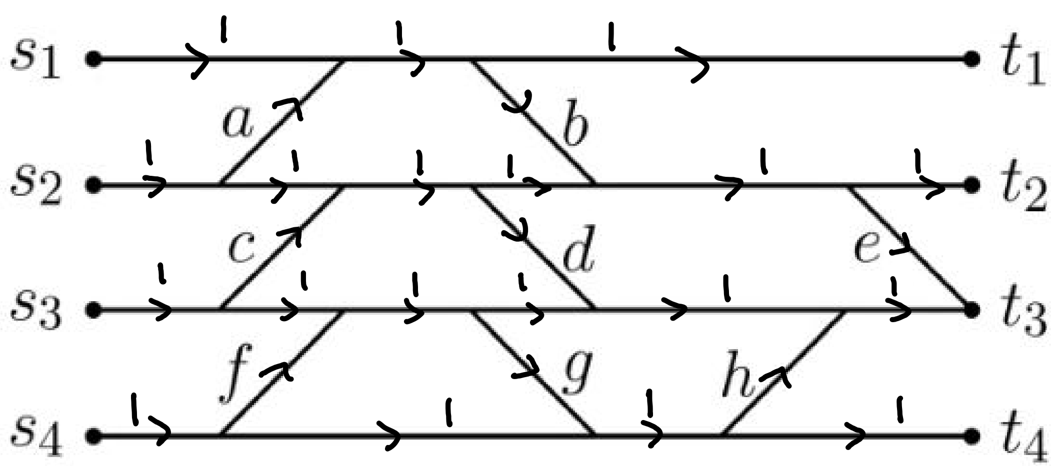
a) Determinants

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

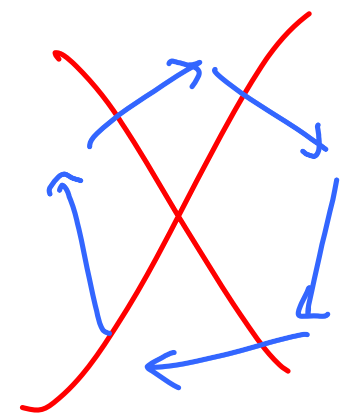
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Objects we need to know about:

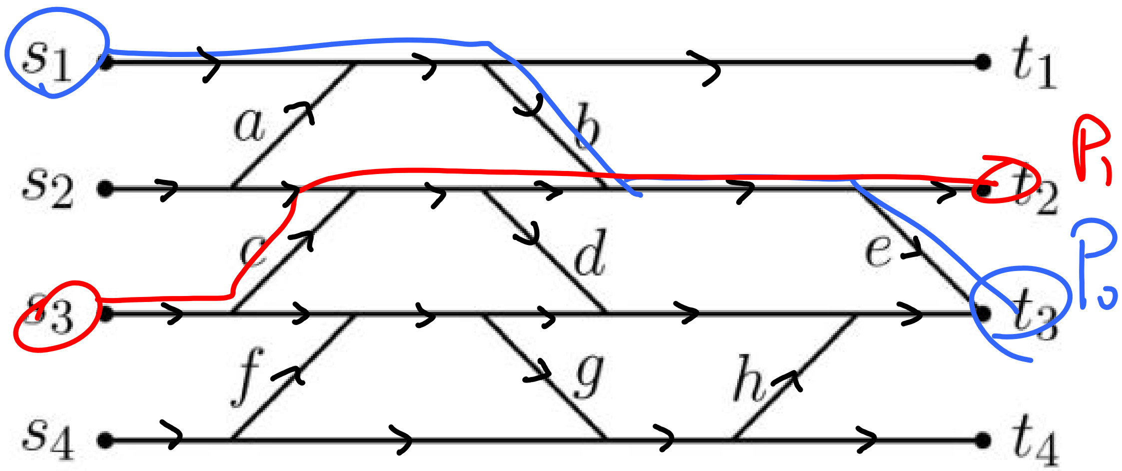
b) Networks (weighted)



acyclic



Paths in a network



$$wt(P) = \prod_{e \in P} wt(e)$$

$$wt(P_0) = be$$

$$wt(P_1) = c$$

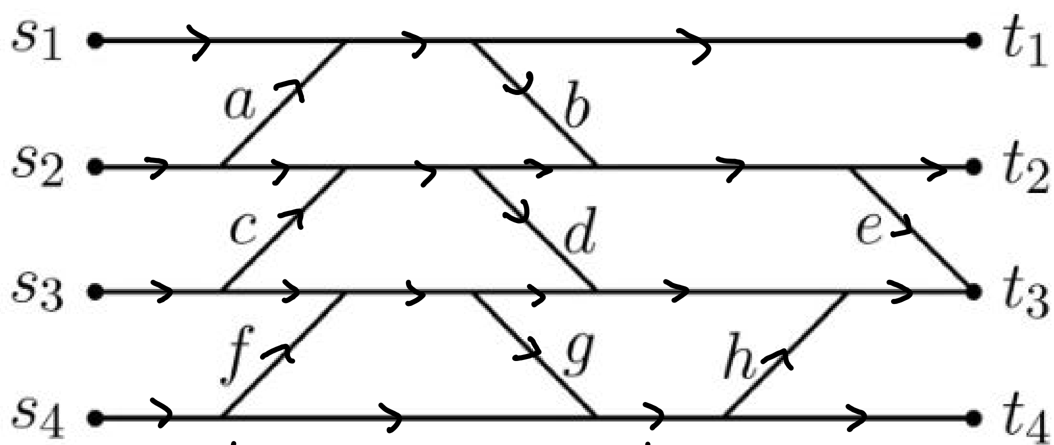
$\mathcal{P} = (P_0, P_1)$, $\quad wt(\mathcal{P}) = be \cdot c$

The weighted path matrix of a network

Let $W=(w_{ij})$ be the matrix whose entries are:

$$w_{ij} = \sum_{P: i \to j} wt(P)$$

Example:

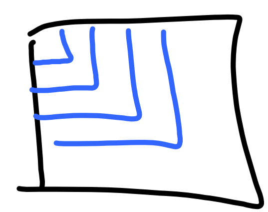


$$
\begin{array}{c c}
 & \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\
\begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} &
\begin{pmatrix}
1 & b & be & 0 \\
a & 1+ab & d+e+abe & 0 \\
0 & c & 1+cd+gh+ce & g \\
0 & 0 & f+h+fgh & 1+fg
\end{pmatrix}
\end{array}
$$

Some nice classes of matrices which arise as weighted path matrices:

- Totally positive matrices
  (all square subdeterminants are >0)

- Positive definite matrices
  (leading principal minors are >0).



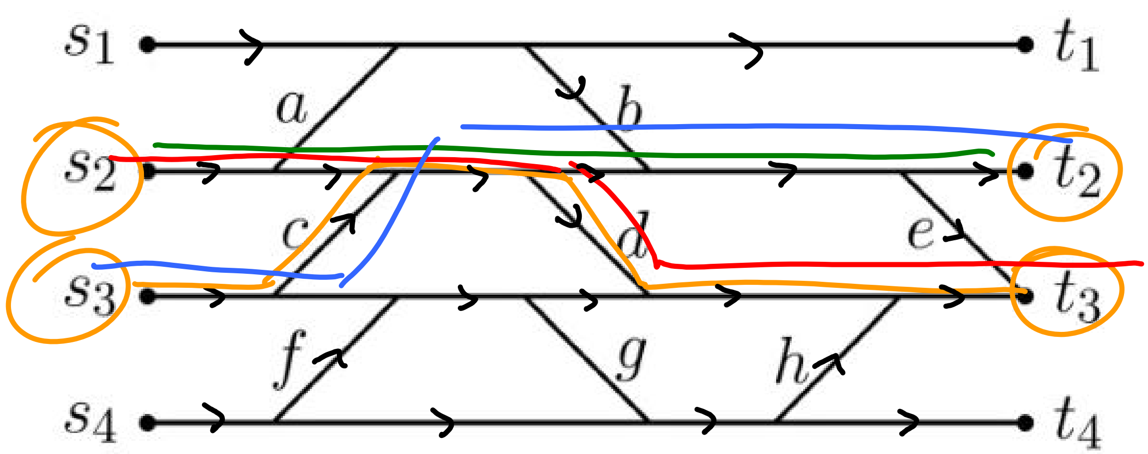
Theorem (Lindström, Gessel–Viennot)

$\boxed{\text{ACYCLIC CASE}}$

Let $W_{A,B}$ be the square submatrix of W whose rows are indexed by A and whose columns are indexed by B.

Then $\det W_{A,B}$ "counts" families of non-intersecting paths connecting A to B.
(I.e. each term is the weight of a NI path family.)

Examples:



$$\det(W) = 1$$

"Sign-reversing involution"

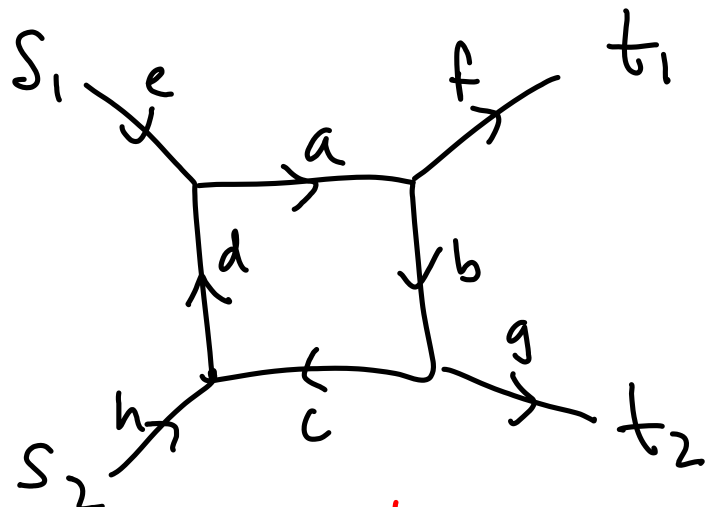
$$\det(W_{23,34}) = e \cdot g + d \cdot g + abe \cdot g$$

$$\begin{array}{c|cc} & 2 & 3 \\ \hline 2 & \bullet & \\ 3 & & \bullet \end{array} \qquad \begin{array}{c|cc} & 2 & 3 \\ \hline 2 & & \bullet \\ 3 & \bullet & \end{array}$$

"Proofs from the book" (Aigner Ziegler)

__Question__: What happens if we try to make the weighted path matrix of a network with one or more directed cycles?

Eg:



$s_1 \rightsquigarrow t_1$?

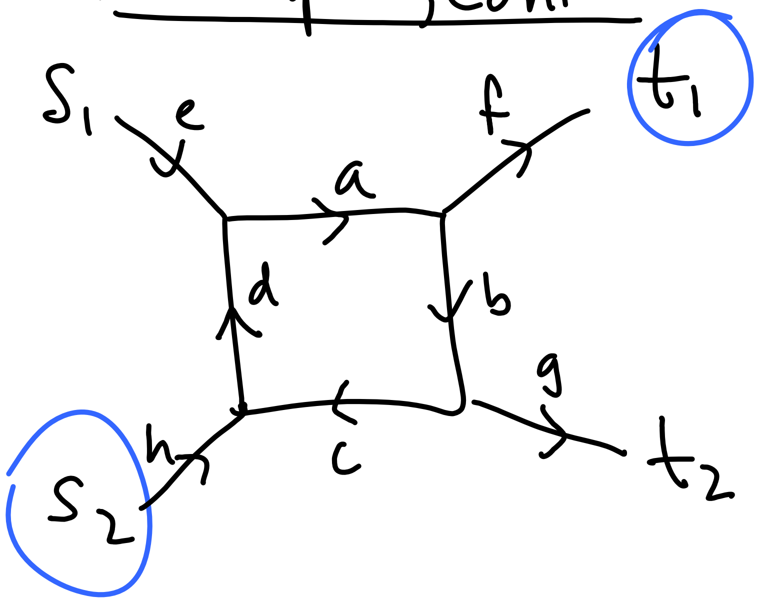
$$eaf + eaf \cdot abcd + eaf(abcd)^2 + \dots$$

$$= eaf\left(1 + abcd + (abcd)^2 + \dots\right)$$

$$= \frac{eaf}{1 - abcd}$$

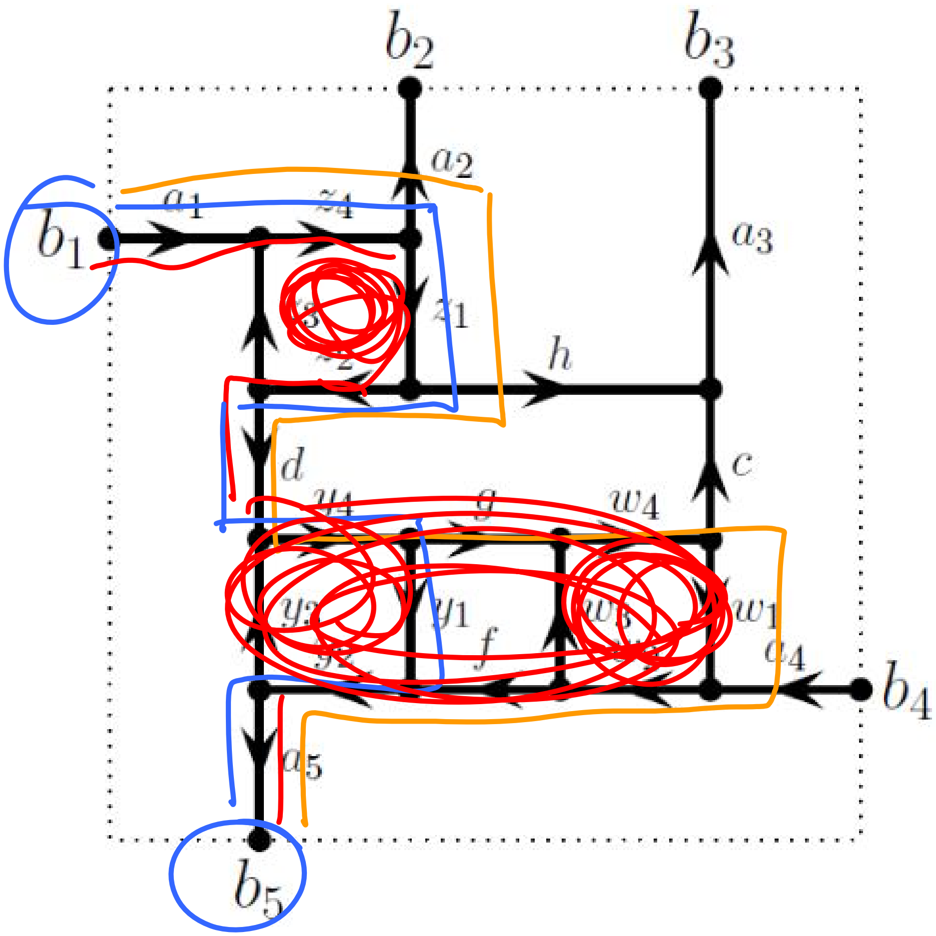
$$1 + x + x^2 + \dots = \frac{1}{1-x}$$

Example, cont:



What if we try a harder one?

Paths from b_1 to b_5?



?

Theorem (Talaska 2011):

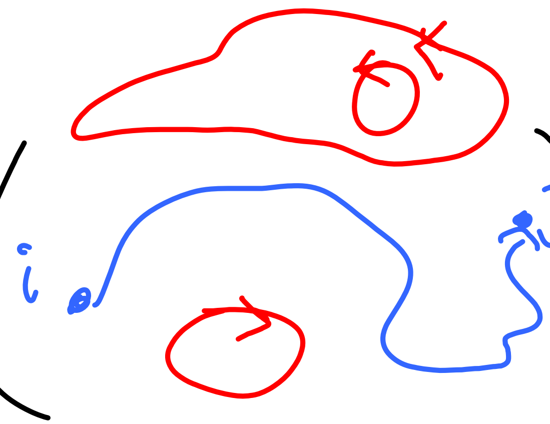
Suppose W is the weighted path matrix of a network (weights are formal variables).

Then a) The formal power series entries of the matrix can be written as

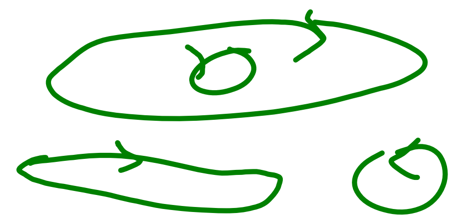
$$W_{ij} = \frac{\displaystyle\sum_{C,\, P: i \rightsquigarrow j} wt(P) \cdot sgn(C)\, wt(C)}{\displaystyle\sum_{C'} sgn(C')\; wt(C')} \qquad (C \cap P = \emptyset)$$

a) in picture form

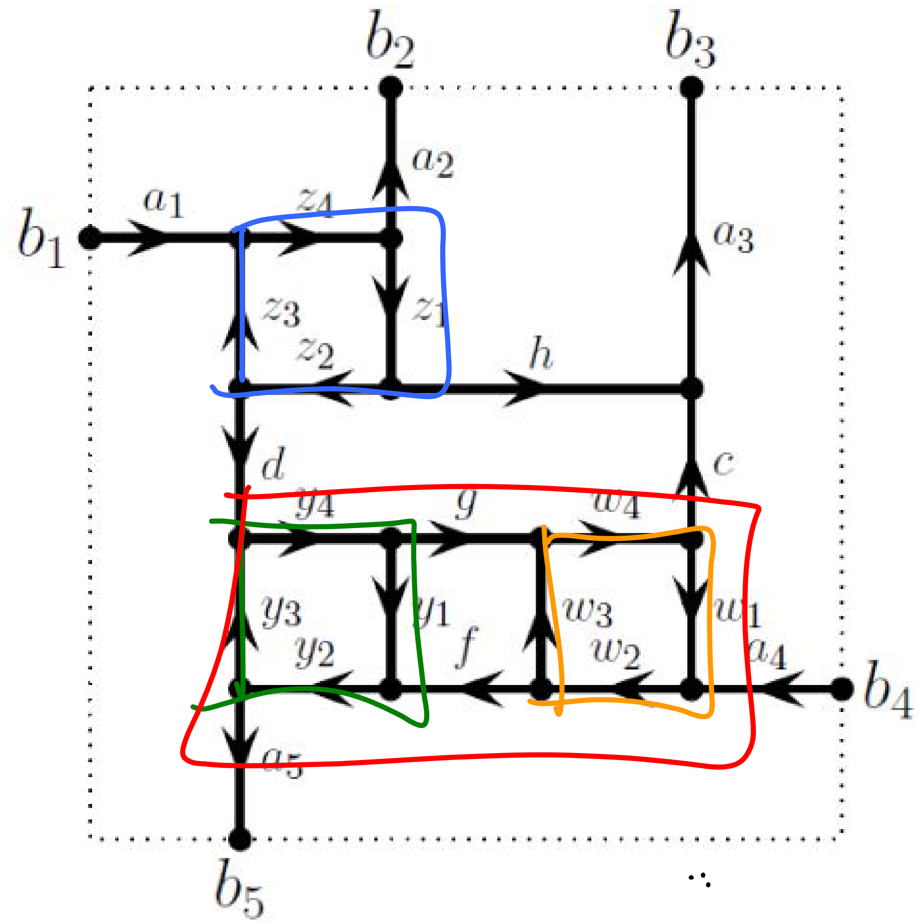
$W_{ij} =$

$\sum \pm \text{wt} \Big($  $\Big)$

divided by

$\sum \pm \text{wt} \Big($  $\Big)$

Example:



$$\begin{array}{c} \\ b_1 \\ b_4 \end{array}\begin{array}{c} \begin{array}{ccc} b_2 & b_3 & b_5 \end{array} \\ \begin{bmatrix} & & * \\ & & \end{bmatrix} \end{array}$$

3terms

$$* = \Big(\mathrm{wt}(\sim)\,\big(1 \pm \mathrm{wt}(\cdots\cdots)\big) \pm \mathrm{wt}(\sim) \Big)$$

$1 \pm \sim \pm \sim \pm \sim \pm \sim$

$\pm \approx \pm \approx \pm \approx \pm \approx$

$\pm \overset{\sim}{\approx}$

10 terms

Theorem, cont:

and b) if $W_{A,B}$ is any square submatrix,

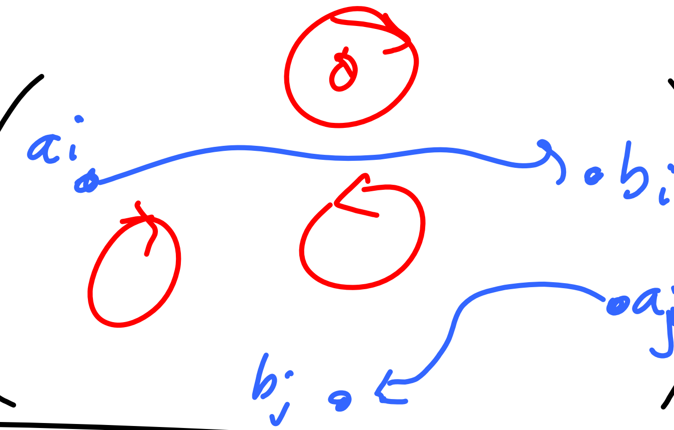
$$\det W_{A,B} = \frac{\displaystyle\sum_{C,\ P : A \rightsquigarrow B} \operatorname{sgn}(P)\,\operatorname{wt}(P)\,\operatorname{sgn}(C)\,\operatorname{wt}(C)}{\displaystyle\sum_{C'} \operatorname{sgn}(C)\,\operatorname{wt}(C)}.$$

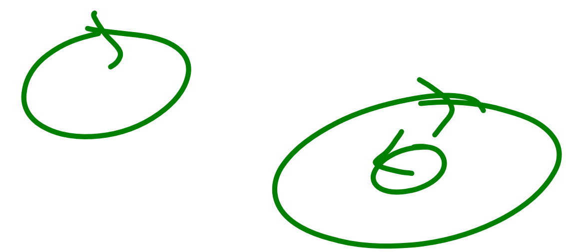
C, C' = collections of non intersecting cycles

$C \cap P = \emptyset$, P = collection of N-I paths, $A \rightsquigarrow B$

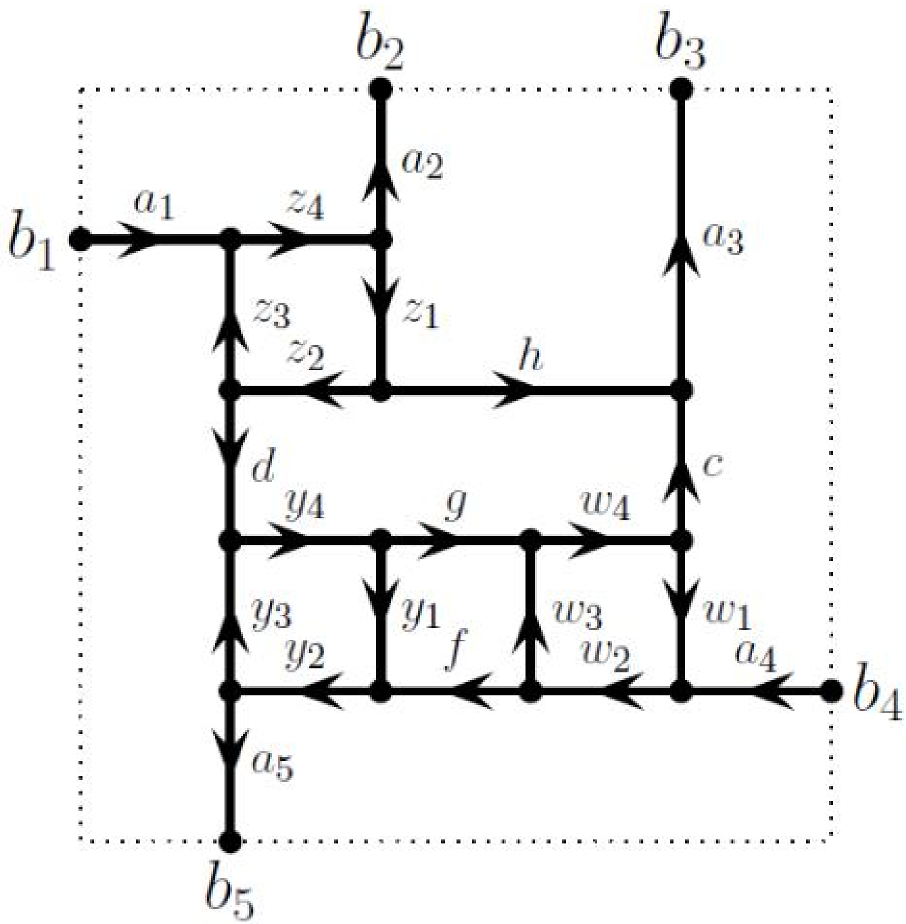
b) in picture form:

$$\det W_{A,B} = \frac{\sum \pm \text{wt}\left(\quad \right)}{\sum \pm \text{wt}\left(\quad \right)}$$





Example:



$$\begin{array}{cc} & \begin{array}{ccc} b_2 & b_3 & b_5 \end{array} \\ \begin{array}{c} b_1 \\ b_4 \end{array} & \left[\begin{array}{ccc} & * & * \\ & * & * \end{array}\right] \end{array}$$

$$\det \boxed{\begin{matrix} * & * \\ * & * \end{matrix}}$$

$$=$$