Mutual Tangent Lines

This problem illustrates a method you can use to find lines that are tangent to *two* curves.

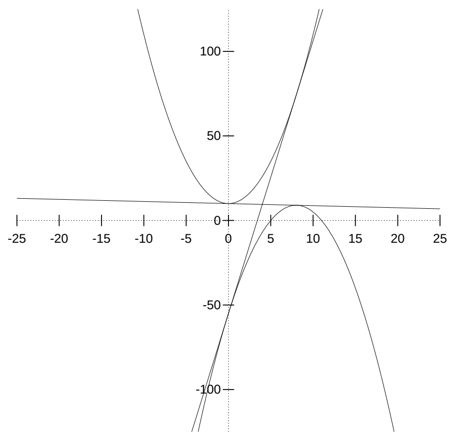
Consider the quadratic functions

$$f(x) = x^2 + 10$$

and

$$g(x) = -(x-8)^2 + 9$$

If we sketch their graphs on the same axes, we can see that there are two lines that are tangent to both curves:



Let's find the equations of those lines. Consider either line.

Suppose it is tangent to $y = f(x)$ at the point (a, b). Suppose it is tangent to $y = g(x)$ at the point (c, d).

This looks like we are introducing *four* variables: a, b, c, and d. But in fact there are only really two: since (a, b) is on the graph of $y = f(x)$, we have

$$b = f(a)$$

Similarly,

$$d = g(c)$$

So we just need to find the values of a and c.

If we could write two equations involving a and c, we'd be in business (we'd have "two equations in two unknowns", and that's a good place to be algebraically).

The way to get the two equations is to write the slope of the tangent line in several ways.

First, since the line passes through (a, b) and (c, d), we know the slope of the line is

$$m = \frac{d-b}{c-a}$$

Also, since the line is tangent to $y = f(x)$, we have

$$m = f'(a)$$

and since the line is tangent to $y = g(x)$, we also have

$$m = g'(c).$$

Putting these together, we get two equations:

$$\frac{d-b}{c-a} = f'(a) \tag{1}$$

and

$$f'(a) = g'(c) \tag{2}$$

Let's see what each equation says.

Equation (1) says

$$\frac{g(c)-f(a)}{c-a} = 2a$$

so

$$g(c) - f(a) = 2a(c-a).$$

Using the definitions of $g(x)$ and $f(x)$, this equation becomes

$$-(c-8)^2 + 9 - (a^2+10) = 2a(c-a). \tag{3}$$

Let's leave that equation alone for a moment and see what equation (2) says.

We know

$$f'(x) = 2x$$

and

$$g'(x) = -2(x-8)$$

so equation (2) says

$$2a = -2(c - 8)$$

so $a = 8 - c$, or $c = 8 - a$.

Using this in equation (3), we get

$$-(8 - a - 8)^2 + 9 - (a^2 + 10) = 2a(8 - a - a)$$

This is an equation in just the variable a, and it's quadratic, so at this point we can tell that we'll be able to find a. Continuing with the last equation:

$$\begin{aligned} -a^2 - a^2 + 9 - 10 &= 16a - 4a^2 \\ -2a^2 - 1 &= 16a - 4a^2 \\ 2a^2 - 16a - 1 &= 0 \end{aligned}$$

Applying the quadratic formula, we get

$$a = \frac{16 \pm \sqrt{16^2 - 4(2)(-1)}}{4} = \frac{16 \pm \sqrt{264}}{4} = 8.06201920231798 \text{ or } -0.06201920231798.$$

These are our two values of a. From these, we get the slopes of the tangent lines, and then we can get the equations of the tangent lines:

$$y = 16.1240384046359(x - 8.06201920231798) + 74.9961536185438$$

and

$$y = -0.12403840463596(x + 0.12403840463596) + 10.0038463814561585$$

This method will work to find the mutual tangent lines of any two quadratic functions.