Writing Problem #2 - Math 125 Honors - Winter Quarter 2009

1. Let $0 < x \leq 1$, and $f(x) = \frac{1}{1+x^2}$. Show that

$$f(x) = 1 - x^2 f(x).$$

2. Use the last fact to show that

$$f(x) = 1 - x^2 + x^4 f(x)$$

$$f(x) = 1 - x^2 + x^4 - x^6 f(x)$$

   and

$$f(x) = 1 - x^2 + x^4 - x^6 + x^8 f(x).$$

3. Show that, for any positive integer $n \geq 1$,

$$1 - x^2 + x^4 - x^6 + \cdots - x^{4n-2} < f(x) < 1 - x^2 + x^4 - x^6 + \cdots + x^{4n}.$$

4. Argue that

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots - \frac{x^{4n-1}}{4n-1} < \tan^{-1}(x) < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n+1}$$

   and that $\tan^{-1}(x)$ can be approximated by

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots - \frac{x^{4n-1}}{4n-1}$$

   with an error of less than

$$\frac{x^{4n+1}}{4n+1}.$$

5. Argue that we can, thus, write

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

   and if the sum on the right is truncated, the error is less than the magnitude of the first truncated term.

6. Conclude the following series representation (known as the Leibniz-Gregory formula):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

   How many terms of this series would you have to use to achieve an estimate of π accurate to 6 decimal places?