

Hints and Partial Solutions to Practice Integrals

1. $\int \sin^4 x \cos^3 x \, dx$

$$= \int \sin^4 x \cos^2 x \cos x \, dx = \int (\sin^4 x) (1 - \sin^2 x) \cos x \, dx$$

Now let $u = \sin x$.

2. $\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x - 3)^2}}$. Let $x - 3 = 3 \sin u$ so the integral is

$$\int \frac{3 \cos u \, du}{\sqrt{9 - 9 \sin^2 u}} = \int \frac{\cos u}{\cos u} \, du = u + C = \sin^{-1} \left(\frac{1}{3}x - 1 \right) + C.$$

3. $\int \frac{\cos x \, dx}{\sqrt{1 + \sin x}}$ Let $u = 1 + \sin x$. Then the integral is $\int \frac{du}{\sqrt{u}} = \dots$

4. $\int \frac{dx}{\sqrt{(a^2 + x^2)^3}}$

Let $x = a \tan u$. Then the integral is

$$\int \frac{a \sec^2 u}{(a^2 \sec^2 u)^{\frac{3}{2}}} \, du = \int \frac{a \sec^2 u}{a^3 \sec^3 u} \, du = \frac{1}{a^2} \int \frac{1}{\sec u} \, du = \frac{1}{a^2} \int \cos u \, du = \dots$$

5. $\int \ln \sqrt{1 + x^2} \, dx = \frac{1}{2} \int \ln(1 + x^2) \, dx$. Use integration by parts with $u = \ln(1 + x^2)$, $dv = dx$, so we get

$$\frac{1}{2} \int \ln(1 + x^2) \, dx = \frac{1}{2} \left(x \ln(1 + x^2) - \int \frac{2x^2}{1 + x^2} \, dx \right)$$

Use the partial fractions method on this last integral, with

$$\frac{2x^2}{1 + x^2} = 2 - \frac{2}{x^2 + 1}.$$

6. $\int \frac{\sin^{-1} x \, dx}{\sqrt{1 - x^2}}$ Let $u = \sin^{-1} x$ so $du = \frac{dx}{\sqrt{1 - x^2}}$. Then the integral is

$$\int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\sin^{-1} x)^2 + C.$$

7. $\int \frac{(3x - 7) \, dx}{(x - 1)(x - 2)(x - 3)}$ Use the partial fractions method to find constants A , B , and C so that

$$\frac{3x - 7}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}$$

$$8. \int e^{\ln \sqrt{x}} dx = \int \sqrt{x} dx = \dots$$

$$9. \int x^2 e^x dx \text{ Use integration by parts twice.}$$

$$10. \int \frac{\tan x dx}{\cos^2 x} = \int \tan x \sec^2 x dx. \text{ Let } u = \tan x \dots$$

$$11. \int \frac{\cos \sqrt{x} dx}{\sqrt{x}} \text{ Let } u = \sqrt{x}.$$

$$12. \int \frac{(x+1) dx}{x^2(x-1)} \text{ Use the partial fractions method to show that}$$

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

and go from there.

$$13. \int \tan^{-1} 10x dx \text{ Let } \theta = 10x. \text{ Then the integral equals}$$

$$\frac{1}{10} \int \tan^{-1} \theta d\theta.$$

Use integration by parts with $u = \tan^{-1} \theta$ and $dv = d\theta$ to get

$$\frac{1}{10} \left(\theta \tan^{-1} \theta - \int \frac{\theta d\theta}{\theta^2 + 1} \right) \dots$$

$$14. \int \frac{dx}{(x^2 - 1)^{3/2}} \text{ Let } x = \sec \theta. \text{ Then the integral equals}$$

$$\int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} = \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \int \frac{\cos \theta}{\sin^2 \theta} d\theta.$$

Now let $w = \sin \theta$.

$$15. \int x^{2/3} (x^{5/3} + 1)^{2/3} dx \text{ Let } u = x^{5/3} \text{ so the integral becomes}$$

$$\frac{3}{5} \int (u+1)^{2/3} du = \dots$$

$$16. \int \tan^3 3x \sec 3x dx = \int \tan^2 3x \sec 3x \tan 3x dx = \int (\sec^2 3x - 1) \sec 3x \tan 3x dx.$$

Let $u = \sec 3x$. Then the integral becomes

$$\frac{1}{3} \int (u^2 - 1) du = \dots$$

17. $\int e^x \cos 2x \, dx$ Use integration by parts twice.

18. $\int \frac{4x + 1}{x^3 + 4x} \, dx$

Use partial fractions to find A, B and C so that

$$\frac{4x + 1}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} = \frac{A}{x} + \frac{Bx}{x^2 + 4} + \frac{C}{x^2 + 4}$$

then integrate, using a trig. substitution on the last term.

19. $\int \frac{dx}{x^2 + 4x + 8} = \int \frac{dx}{(x + 2)^2 + 4}$ Use a trig. substitution.

20. $\int x^2 \sin x \, dx$ Use integration by parts twice.

21. $\int \sin^4 x \cos^2 x \, dx = \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \left(\frac{1}{2}(1 + \cos 2x)\right) \, dx = \dots$

22. $\int \frac{x^5 \, dx}{\sqrt{1 + x^2}}$ Use $x = \tan \theta$ to change the integral to

$$\int \tan^5 \theta \sec \theta \, d\theta = \int \tan^4 \theta \sec \theta \tan \theta \, d\theta = \int (\sec^2 \theta - 1)^2 \sec \theta \tan \theta \, d\theta$$

Now let $u = \sec \theta \dots$

23. $\int \frac{\cos x \, dx}{\sin^3 x - \sin x}$ Let $u = \sin x$. The integral becomes

$$\int \frac{du}{u^3 - u} = \int \frac{du}{u(u^2 - 1)} = \int \frac{du}{u(u + 1)(u - 1)}$$

Now use partial fractions.

24. $\int x \tan^2 x \, dx = \int x(1 - \sec^2 x) \, dx = \int x \, dx - \int x \sec^2 x \, dx$. Use integration by parts on the second integral.

25. $\int \tan^2 x \sec^2 x \, dx$ Let $u = \tan x$ so that the integral equals

$$\int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C.$$

26. $\int x \ln \sqrt{x + 2} \, dx = \frac{1}{2} \int x \ln(x + 2) \, dx$. Use integration by parts.

27. $\int \frac{x \, dx}{x^4 - 16} = \int \frac{x}{(x^2 - 4)(x^2 + 4)} \, dx = \int \frac{x}{(x - 2)(x + 2)(x^2 + 4)} \, dx$. Use partial fractions.

28. $\int \frac{x^3 + x^2}{x^2 + x - 2} \, dx = \int x + \frac{2x}{(x + 2)(x - 1)} \, dx$. Use partial fractions.

$$29. \int \frac{dx}{x^{1/5}} = \int x^{-\frac{1}{5}} dx = \frac{5}{4}x^{\frac{4}{5}} + C.$$

30. $\int \sin x \cos 2x dx$ Use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to write $\cos 2x = 2 \cos^2 x - 1$ and the integral becomes

$$\int \sin x(2 \cos^2 x - 1) dx = 2 \int \cos^2 x \sin x dx - \int \sin x dx = \dots$$

$$31. \int \frac{dx}{x^4 + 4x^2 + 3} = \int \frac{dx}{(x^2 + 3)(x^2 + 1)}.$$

Use the partial fractions method to write

$$\frac{1}{(x^2 + 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{x^2 + 1}$$

Find the constants and integrate.

32. $\int \frac{x^3 dx}{\sqrt{16 - x^2}}$ Let $x = 4 \sin \theta$ so the integral becomes

$$\int \frac{64 \sin^3 \theta}{4 \cos \theta} 4 \cos \theta d\theta = 64 \int \sin^3 \theta d\theta = \dots$$

33. $\int \sin^3 2x \cos^3 2x dx = \int \sin^3 2x \cos^2 2x \cos 2x dx = \int (\sin^3 2x)(1 - \sin^2 2x) \cos 2x dx.$

Let $u = \sin 2x \dots$

34. $\int \frac{dx}{x(2 + \ln x)}$ Let $u = \ln x$ so the integral becomes

$$\int \frac{du}{2 + u} = \ln |2 + u| + C = \ln |2 + \ln x| + C.$$

35. $\int \frac{\cot x dx}{\ln(e \sin x)} = \int \frac{\cos x dx}{\sin x \ln(e \sin x)}$. Let $u = \sin x$ so the integral becomes

$$\int \frac{du}{u \ln(eu)}.$$

Now let $v = \ln eu$ so the integral becomes

$$\int \frac{dv}{v} = \ln |v| + C = \ln |\ln(e \sin x)| + C$$

36. $\int \frac{\sec^2 x dx}{\sec^2 x - 3 \tan x - 1} = \int \frac{\sec^2 x dx}{\tan^2 x - 3 \tan x}$. Let $u = \tan x$ to get

$$\int \frac{du}{u^2 - 3u} = \int \frac{du}{u(u - 3)}$$

and use partial fractions.