Fixed points of linear to linear rational functions

Suppose the function

$$f(x) = \frac{Ax + B}{x + C}$$

has fixed points x_1 and x_2 (with $x_1 \neq x_2$). In other words,

$$f(x_1) = x_1 \text{ and } f(x_2) = x_2.$$

Let's assume for now that x_1 and x_2 are not zero.

Then

$$f(x_1) = x_1 = \frac{Ax_1 + B}{x_1 + C}$$

and

$$f(x_2) = x_2 = \frac{Ax_2 + B}{x_2 + C}$$

so that

$$x_1^2 + x_1 C = Ax_1 + B \tag{1}$$

and

$$x_2^2 + x_2 C = Ax_2 + B \tag{2}$$

Multiplying equation (1) by x_2 and equation (2) by x_1 yields

$$x_1^2 x_2 + x_1 x_2 C = Ax_1 x_2 + Bx_2 \tag{3}$$

and

$$x_1 x_2^2 + x_1 x_2 C = Ax_1 x_2 + Bx_1 \tag{4}$$

Subtracting equation (4) from equation (3) yields

$$x_1^2 x_2 - x_1 x_2^2 = B(x_2 - x_1)$$

so that

$$B = \frac{x_1 x_2 (x_1 - x_2)}{x_2 - x_1} = -x_1 x_2$$

Now this is all assuming that x_1 and x_2 are not zero. Suppose 0 is a fixed point of f. Then

$$f(0) = 0 = \frac{B}{C}$$

from which we can immediately conclude that $B = 0$, so in this case $B = -x_1 x_2$.

Thus B is determined by the fixed points of the function.

What this means is that if we are seeking a linear-to-linear rational function with given fixed points, we are *not* free to pick B to be whatever we want. We are free to pick A or C to be whatever we want, however.

So, a good procedure for finding a linear-to-linear rational function with specified fixed points x_1 and x_2 is to first pick A or C, and then solve for the other variables. This will always work, except we do need to be careful to not pick C to be $-x_1$ or $-x_2$, or pick A equal to x_1 or x_2 (why?).