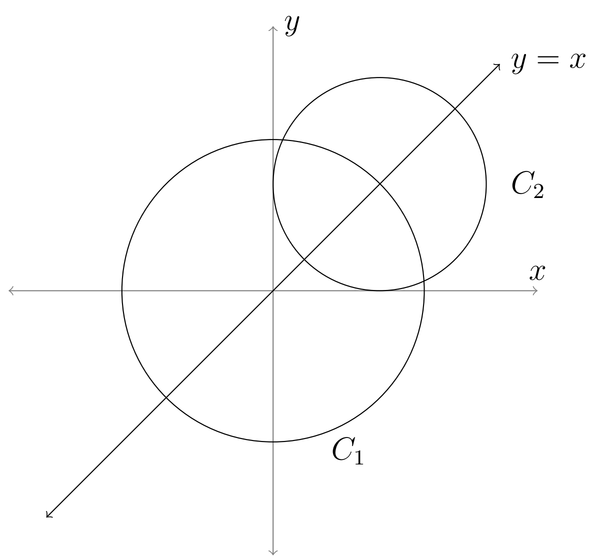
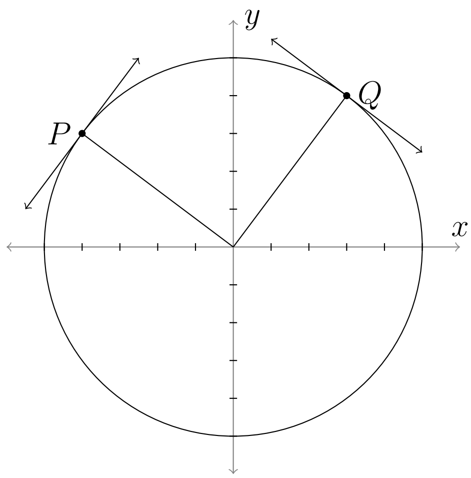
# Worksheet for Week 1: Circles and lines

This worksheet is a review of circles and lines, and will give you some practice with algebra and with graphing. Also, this worksheet introduces the idea of "tangent lines" to circles. Later on in Math 124, you'll learn how to find tangent lines to many other types of curves.

1. Two circles, called $C_1$ and $C_2$, are graphed below. The center of $C_1$ is at the origin, and the center of $C_2$ is the point in the first quadrant where the line $y = x$ intersects $C_1$. Suppose $C_1$ has radius 2. $C_2$ touches the $x$ and $y$ axes each in one point. What are the equations of the two circles?

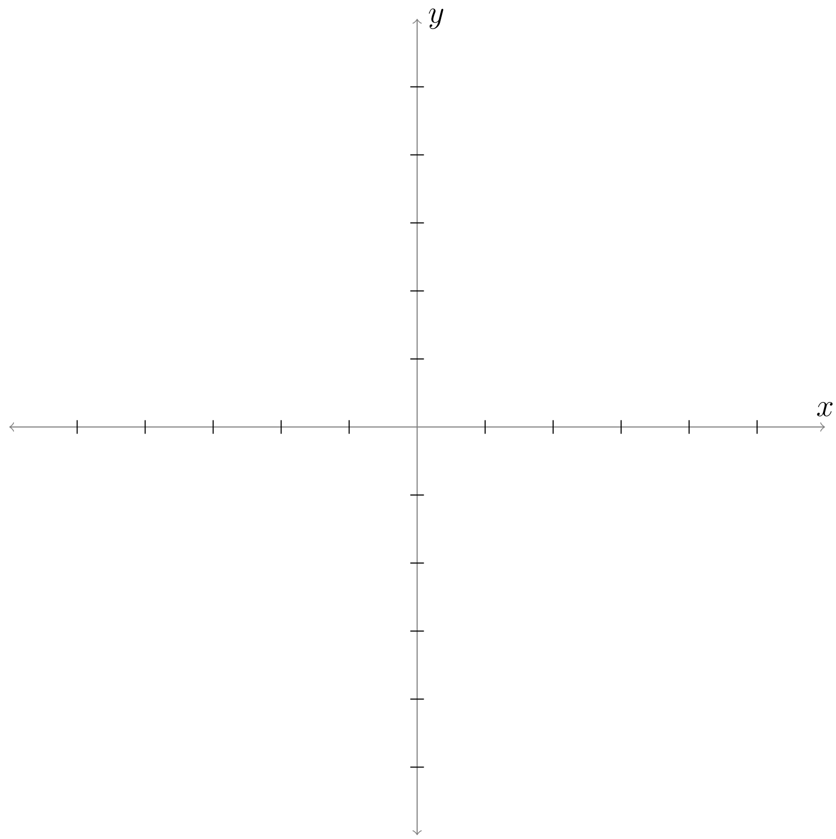


2. Let $C$ be the circle of radius 5 centered at the origin. The **tangent line** to $C$ at a point $Q$ is the line through $Q$ that's perpendicular to the radial line connecting $Q$ to the center. (See picture.) Use this information to find the equations of the tangent lines at $P$ and $Q$ below.




**Note:** Later in Math 124, you'll learn how to find tangent lines to curves that are not circles!

3. Sketch the circle of radius 2 centered at $(3, -3)$ and the line $L$ with equation $y = 2x + 2$. Find the coordinates of all the points on the circle where the tangent line is perpendicular to $L$.

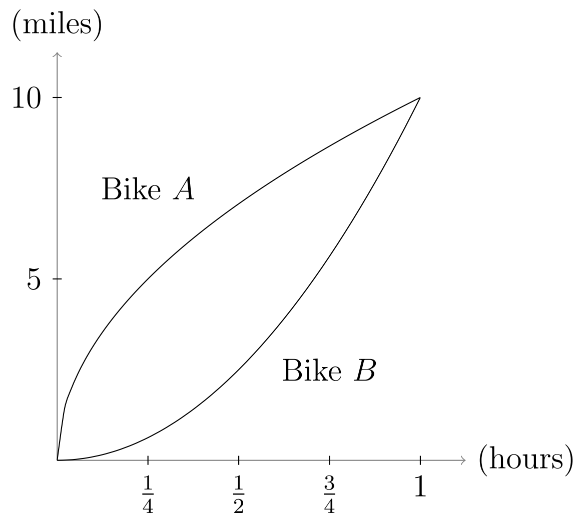


4. Draw the circle with equation $x^2+y^2 = 25$ and the points $P = (-3, -4)$ and $Q = (-8, 0)$. Explain why $P$ is on the circle. Is the line through $P$ and $Q$ tangent to the circle? How do you know?

# Worksheet for Week 2: Graphs and limits

In this worksheet, you'll practice using the graph of an object's position to learn about its velocity. You'll also learn a useful technique for computing limits of certain types of functions at points where the function might not be defined.

1. Consider the graph below, which shows how the positions of two bicycles (called $A$ and $B$) change as time passes. The units of position are miles; the units of time are hours.




(a) Which bike is moving faster at $t = \frac{1}{4}$ (that is, after 15 minutes)? How do you know?

(b) Which bike is moving faster at the end of the ride (at $t = 1$)?

(c) Do the bikes finish the hourlong ride together, or does one bicyclist beat the other? How can you tell?

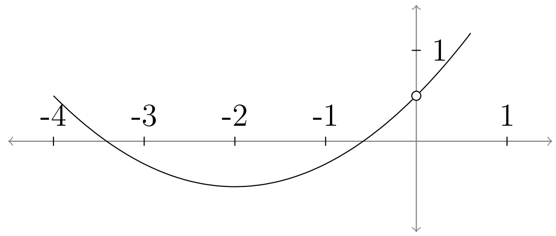
Notice that a steeper curve on the graph corresponds to a higer velocity. A steep curve means that the position is changing a lot in a short amount of time, which means the bike is moving fast.

2. Again, refer to the graph on Page 1 to answer the following questions.
   (a) According to the graph, during **the last half-hour of the bike ride**, when is bike $A$ moving the fastest?

   (b) At about what time does bike $B$ start catching up with bike $A$? That is, when does the distance between bikes $A$ and $B$ start to shrink?

   (c) Do you think there is an instant when bikes $A$ and $B$ are moving at exactly the same velocity? Either estimate that time by looking at the graph, or explain why there can't be such a time.

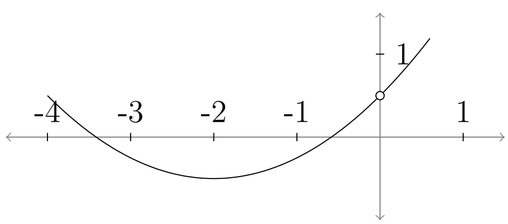
3. Now consider the graph of $f(x)$ below:

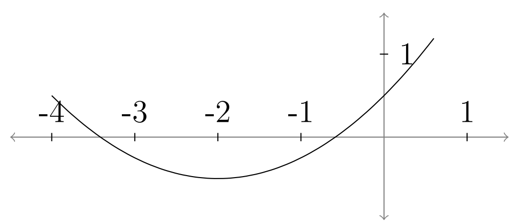



The function $f(x)$ isn't defined at $x = 0$, but you can still find $\lim_{x \to 0} f(x)$ by looking at the graph. What is this limit?

It would be nice not to need to look at a graph to find limits like these, since many functions are very difficult to graph. Fortunately, there is a method that often works for computing such limits. This method, described below, will come up a *lot* in Math 124.

4. Below there are two equations and two graphs. Which equation corresponds to which graph?




$$y = \frac{\frac{1}{4}x^3 + x^2 + \frac{1}{2}x}{x} \qquad\qquad y = \frac{1}{4}x^2 + x + \frac{1}{2}$$

Draw lines connecting each equation to its graph. How do you know your answer is correct?

Now, let $f(x) = (\frac{1}{4}x^3 + x^2 + \frac{1}{2}x)/x$, and let $g(x) = \frac{1}{4}x^2 + x + \frac{1}{2}$. The graphs of $f(x)$ and $g(x)$ above are identical except at $x = 0$: there $g(x)$ is defined and $f(x)$ is not.

5. Why are the graphs of $f(x)$ and $g(x)$ identical except at $x = 0$? (If you have trouble seeing the reason, try plugging in specific non-zero $x$'s. What goes wrong when you try to plug in $x = 0$?)

6. You calculated $\lim\limits_{x \to 0} f(x)$ above in Question 3 by looking at its graph. How does this limit compare to the value $g(0)$?

Let's review: we had a function $f(x)$ that had a hole at $x = 0$, and we hoped to find $\lim\limits_{x \to 0} f(x)$ without needing to refer to a graph. We did this by using a function $g(x)$ that is exactly the same as $f(x)$ except that it *is* defined at $x = 0$. That is, we filled in the gap in the graph of $f(x)$. Then we could just plug 0 into $g(x)$ to find the limit. This method works because the new function $g$ is *continuous*. You will see continuous functions later in class.

7. Find $\lim_{x \to -1} \frac{x^2 - x - 2}{x+1}$. Where is the function not defined? How can you fix it?

# Worksheet for Week 3: Graphs of $f(x)$ and $f'(x)$

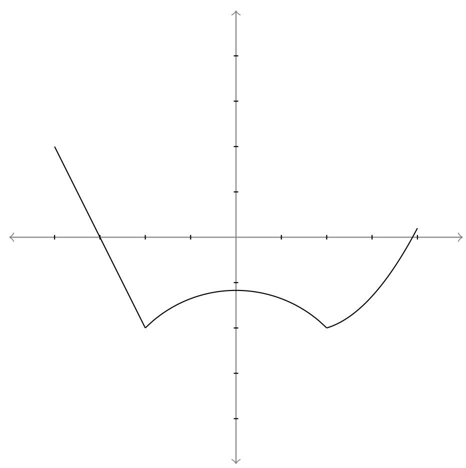
In this worksheet you'll practice getting information about a derivative from the graph of a function, and vice versa. At the end, you'll match some graphs of functions to graphs of their derivatives.

If $f(x)$ is a function, then remember that we define

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If this limit exists, then $f'(x)$ is the slope of the tangent line to the graph of $f$ at the point $(x, f(x))$.

Consider the graph of $f(x)$ below:



1. Use the graph to answer the following questions.
   (a) Are there any values $x$ for which the derivative $f'(x)$ does *not* exist?

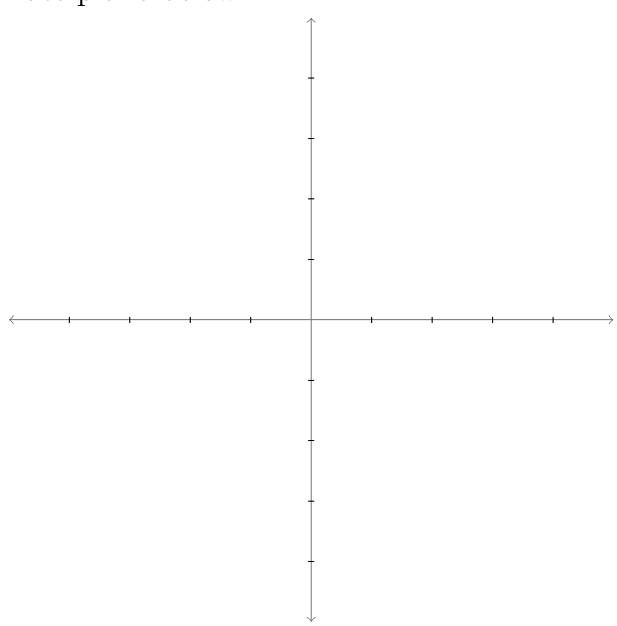
   (b) Are there any values $x$ for which $f'(x) = 0$?

(c) This particular function $f$ has an interval on which its derivative $f'(x)$ is constant. What is this interval? What does the derivative function look like there? Estimate the slope of $f(x)$ on that interval.

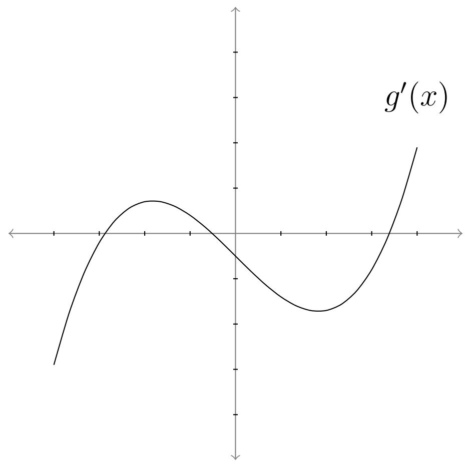
(d) On which interval or intervals is $f'(x)$ positive?

(e) On which interval or intervals is $f'(x)$ negative? Again, sketch a graph of the derivative on those intervals.

(f) Now use all your answers to the questions to sketch a graph of the derivative function $f'(x)$ on the coordinate plane below.



2. Below is a graph of a derivative $g'(x)$. Assume this is the entire graph of $g'(x)$. Use the graph to answer the following questions about the original function $g(x)$.



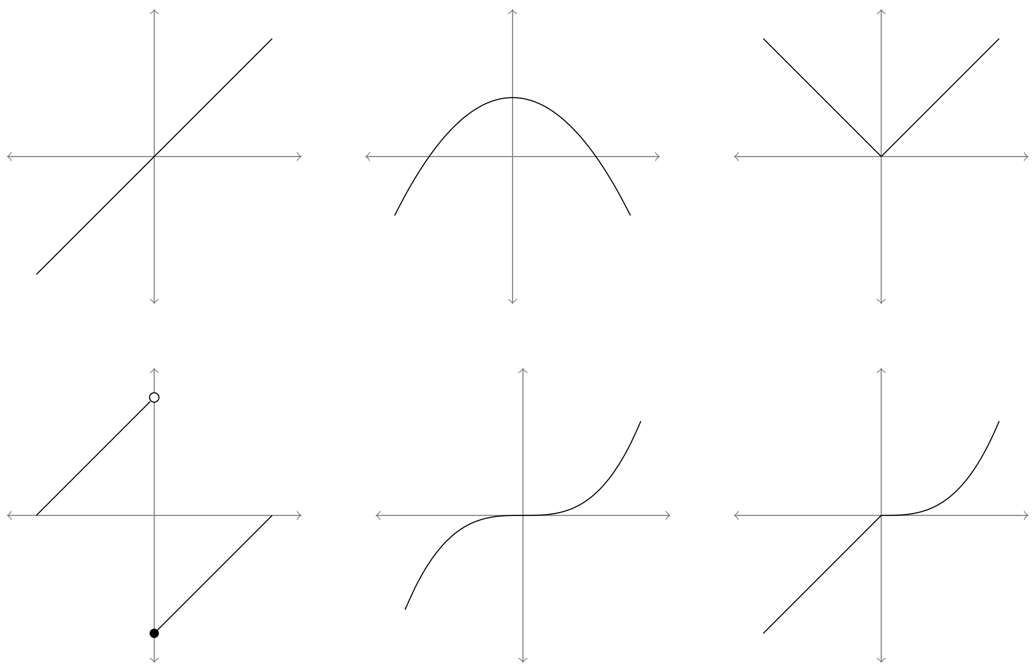

(a) On which interval or intervals is the original function $g(x)$ increasing?

(b) On which interval or intervals is the original function $g(x)$ decreasing?

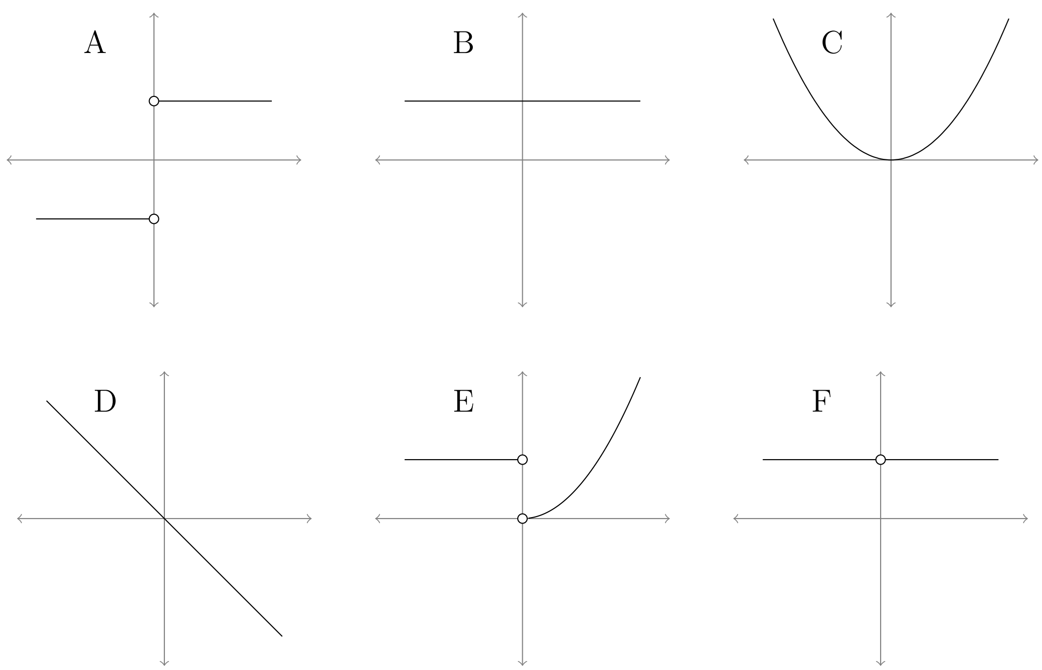
(c) Now suppose $g(0) = 0$. Is the function $g(x)$ ever positive? That is, is there any $x$ so that $g(x) \gneq 0$? How do you know?

3. Six graphs of functions are below, along with six graphs of derivatives. Match the graph of each function with the graph of its derivative.

Original Functions:



Their derivatives:



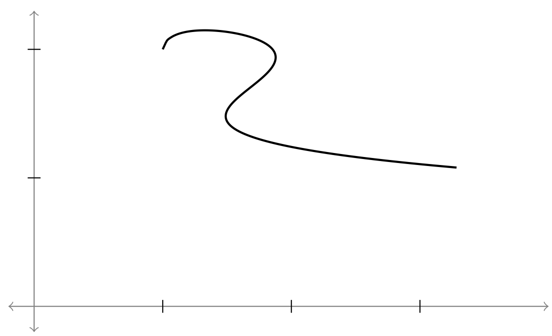
# Worksheet for Week 4: Velocity and parametric curves

In this worksheet, you'll use differentiation rules to find the vertical and horizontal velocities of an object as it follows a parametric curve. You'll also get a preview of how to find tangent lines to parametric curves.

1. A bug is running around on a window. If we put $x$ and $y$ axes on the window, then the bug follows the path given by the parametric equations

$$(x(t), y(t)) = \left(\frac{1}{20}te^t + e^t - t^e,\ t^{2/3} - t + 2\right).$$

The bug's path is shown below, for $0 \leq t \leq 3$.



Use the equations for the bug's path to answer the following questions.

(a) The horizontal velocity of the bug at some time $t$ is given by the derivative $x'(t)$. What is a formula for $x'(t)$?

(b) Now find a formula for $y'(t)$, the vertical velocity of the bug.

(c) Find a point on the path where the vertical velocity $y'$ is zero.

(d) Now look back at the picture of the bug's path on the first page. What can you say about the tangent line to the path at the point you found in Part (c)? Explain why your answer makes sense, given what you know about $y'$ there.

(e) Suppose $(a, b)$ is a point where the horizontal velocity $x'$ is zero, and the vertical velocity $y'$ is *not* zero. What do you predict the tangent line looks like at this point? Why?

2. Now suppose a spider is following the path given by the parametric equations

$$(x(t), y(t)) = \left(t - 2, t^{4/3} - 2t\right).$$

(a) At which point $(a, b)$ is the vertical velocity of the spider equal to 0?

(b) If a curve is given parametrically and if $t_0$ is a time where $x'(t_0) \neq 0$, then the slope of the tangent line to the curve at the point $(x(t_0), y(t_0))$ is given by

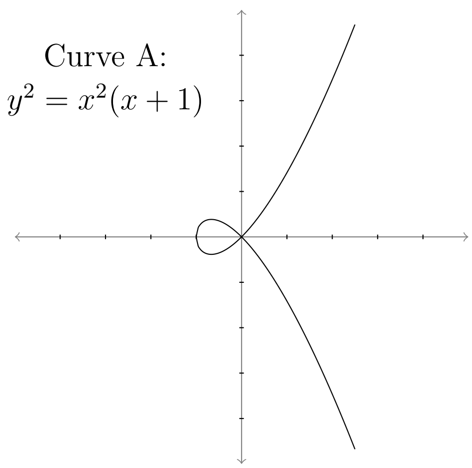
$$\frac{y'(t_0)}{x'(t_0)}.$$

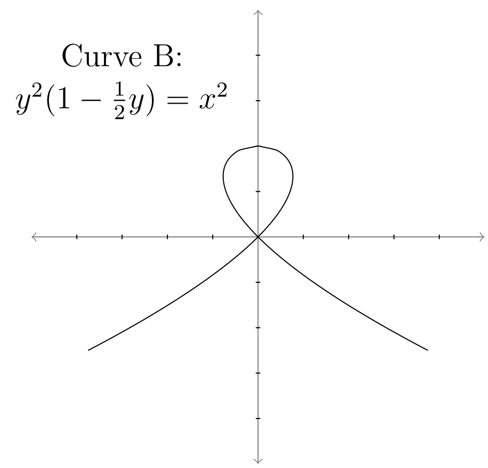
(You will see this again in class.)

Using this information, find the point $(a, b)$ on the spider's path where the tangent line is perpendicular to the line $y = -5x - 12$.

# Worksheet for Week 6: Implicit Differentiation

In this worksheet, you'll use parametrization to deal with curves that have more than one tangent line at a point. Then you'll use implicit differentiation to relate two derivative functions, and solve for one using given information about the other.




1. (a) Use impicit differentiation to find all the points in Curve A with a horizontal tangent line. (Looking at the graph, how many such points should there be?)

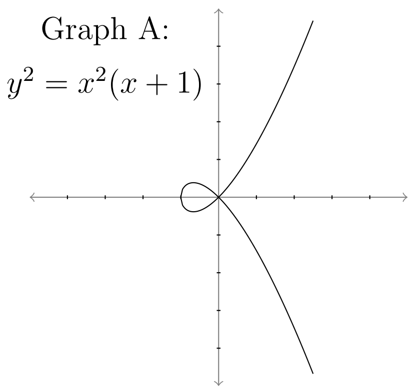
   (b) What about Curve B?

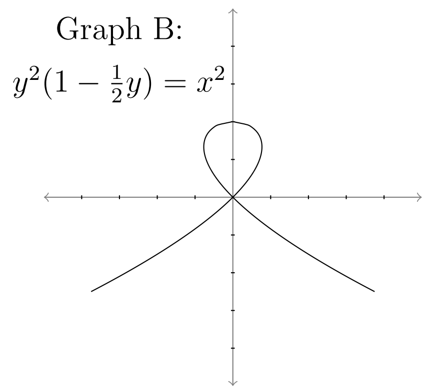
   (c) Try to find $\frac{dy}{dx}$ at the point $(0,0)$ on both graphs. What goes wrong?

2. In this problem you'll look at the curves from Page 1 in a different way.

   Suppose a cat is chasing a ball around on the floor, and its position is described by the parametric equations

$$(x(t), y(t)) = (t^2 - 1, t - t^3).$$




   (a) The cat is following one of the paths from the previous page (reprinted above). Which path does the cat follow? Circle this curve. How do you know it's the right one?

   (b) Draw an arrow on the circled graph above, to indicate in which direction the cat is running.

(c) At which time(s) $t$ does the cat run through the point $(0, 0)$ ?

(d) Remember that it wasn't possible to find $\frac{dy}{dx}$ at $(0, 0)$ using the method on Page 1. But now that the graph has been parametrized, you can do it. What are the tangent line(s) to the parametrized curve $(x(t), y(t))$ at $(0, 0)$?

3. This next question is a new type of problem that you can solve now that you know about implicit differentiation. Suppose a snowball is rolling down a hill, and its radius $r$ is growing at a rate of 1 inch per minute. The volume $V$ of the snowball grows more quickly as the snowball gets bigger. In this question, you'll find the rate of change of the volume, $\frac{dV}{dt}$, at the instant when the radius $r$ is 6 inches.

   (a) First, apply geometry to the situation. Can you think of an equation that relates the variables $r$ and $V$ to each other?

   (b) Now the variables $V$ and $r$ change as time changes, so we can think of them as functions of $t$. Differentiate the equation you came up with in part (a) with respect to $t$.

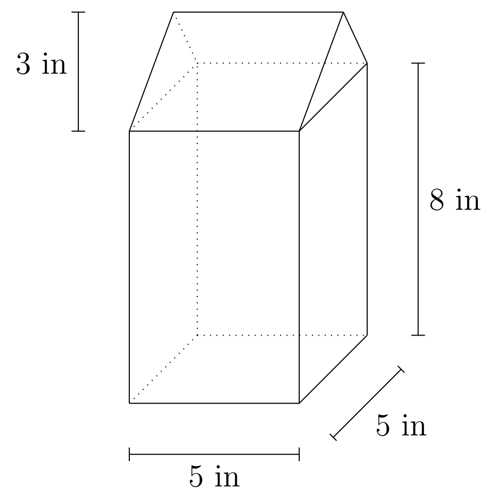
   (c) What is the rate of change of the radius? Use this to simplify your equation from part (b).

   (d) What is the rate of change of $V$ when the radius of the snowball is 6 inches?

# Worksheet for Week 7: Related Rates

This worksheet guides you through some more challenging problems about related rates.

A milk carton is shaped like a tall box with a triangular prism on top. The sides of the top section are isosceles triangles. This particular milk carton has a 5 inch $\times$ 5 inch square base, and is 11 inches tall. (See the picture.)




Suppose you're filling the carton with liquid, at a rate of 10 inches$^3$ per minute. In this problem, you'll figure out the rate of change of the height of the liquid in the carton, at the instant when the carton holds 220 inches$^3$ of liquid.

1. (a) Let $y$ be the height of the liquid in the carton. Then $y$ is a function of time, because $y$ is changing as more liquid pours in. Suppose $y \leq 8$, so that the liquid is all in the rectangular part of the carton. Find a formula for the total volume of liquid in the carton.

   (b) Now suppose $8 \leq y \leq 11$, so that some liquid is in the triangular part of the carton. Find a formula for the total volume of liquid in the carton, in terms of $y$. You might want to break up the volume into two pieces: the volume below the 8-inch line and the volume above the 8-inch line.

(c) Next, suppose that 220 $\text{in}^3$ of liquid is in the carton. How high is the liquid level? That is, what is $y$?

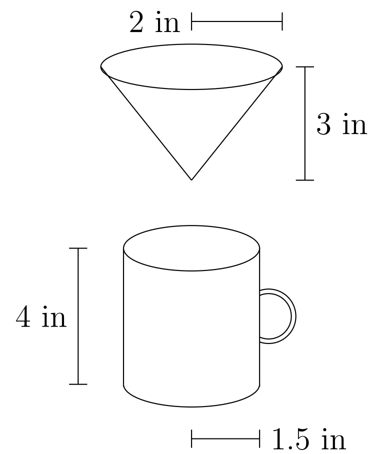
(d) What is $\frac{dy}{dt}$ when 220 $\text{in}^3$ of liquid is in the carton?

(e) When $y$ reaches 8 inches, does $\frac{dy}{dt}$ increase, decrease, or stay the same? You can answer this question by using the formulas you found on Page 1, or by looking at the picture. How do you know your answer is correct?

2. Now suppose a funnel is positioned over a coffee mug.

   The funnel is a cone 3 inches high with a base radius of 2 inches, and the coffee mug is 4 inches tall with a 1.5-inch radius. See the picture at right.

   Suppose the funnel is initially **full** of coffee. Then it starts to drip down into the mug at a constant rate of 0.1 inches$^3$ per second.




   (a) Let $y$ be the height of the coffee in the funnel at any given time, so that just before the dripping starts we have $y = 3$. Since the coffee is draining out of the funnel, $\frac{dy}{dt}$ will be negative. What (approximately) will be the value of $y$ when $\frac{dy}{dt}$ is at its smallest (closest to zero)?

   (b) What (approximately) will $y$ be when $\frac{dy}{dt}$ is biggest (farthest from zero)?

   (c) How much coffee is in the funnel at the very beginning?

   (d) How much coffee is in the funnel **and mug** at some time $t$?

(e) Find a formula, depending on the height $y$ of coffee in the funnel, for the volume of coffee in the funnel.

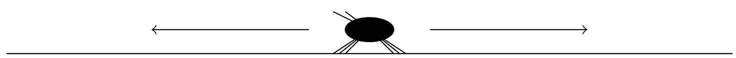
(f) Now suppose the coffee **mug** is one-third full of coffee. How fast is the height of coffee in the funnel changing? In other words, what is $\frac{dy}{dt}$ at that instant?

# Worksheet for Week 10: Sketching Curves

You might have wondered, *why bother learning how to sketch curves using calculus if I can just plug the equation into a computer and see the graph?* But it could happen that you don't actually have a neat formula for the function you're trying to graph. In this worksheet, you'll reconstruct a graph of a function given some data about it. There is no neat formula for the function, but calculus will help you figure out what it looks like anyway!

At the end of the worksheet, there is a problem about maximizing a function on a closed interval, for extra practice.

1. A scientist is watching a bug walk back and forth along a line. Suppose the line has a coordinate $x$, and let $p(t)$ be the continuous function giving the bug's position on the line at time $t$ (in seconds).



The scientist observes the bug's motions and records what she sees:

- At $t = 0$, the bug was located at $x = -0.25$.
- At $t = 5$, the bug passed through the point $x = 0$ for the first and only time.
- As $t \to \infty$, the bug approaches $x = 0$. In other words, $\lim_{t\to\infty} p(t) = 0$.
- The derivative funtion $p'(t)$ — the velocity of the bug — is continuous. For a few seconds at the beginning $p'(t)$ was negative, but then it crossed 0 to become positive at $t = 3.3$. It crossed 0 to become negative again at $t = 6.7$, and remained negative thereafter.
- Also, $p'(t)$ had its maximum value at $t = 5$, and its most negative value at $t = 2.2$ and $t = 7.8$.
- The bug always stayed within 5 units of $x = 0$.

In this problem, you'll figure out how to sketch a graph of the bug's movements on the interval $[0, \infty)$, even though you don't know the formula for $p(t)$! For now, use the information above to answer the following questions. (*After* you answer them all, you'll get to make the sketch.)

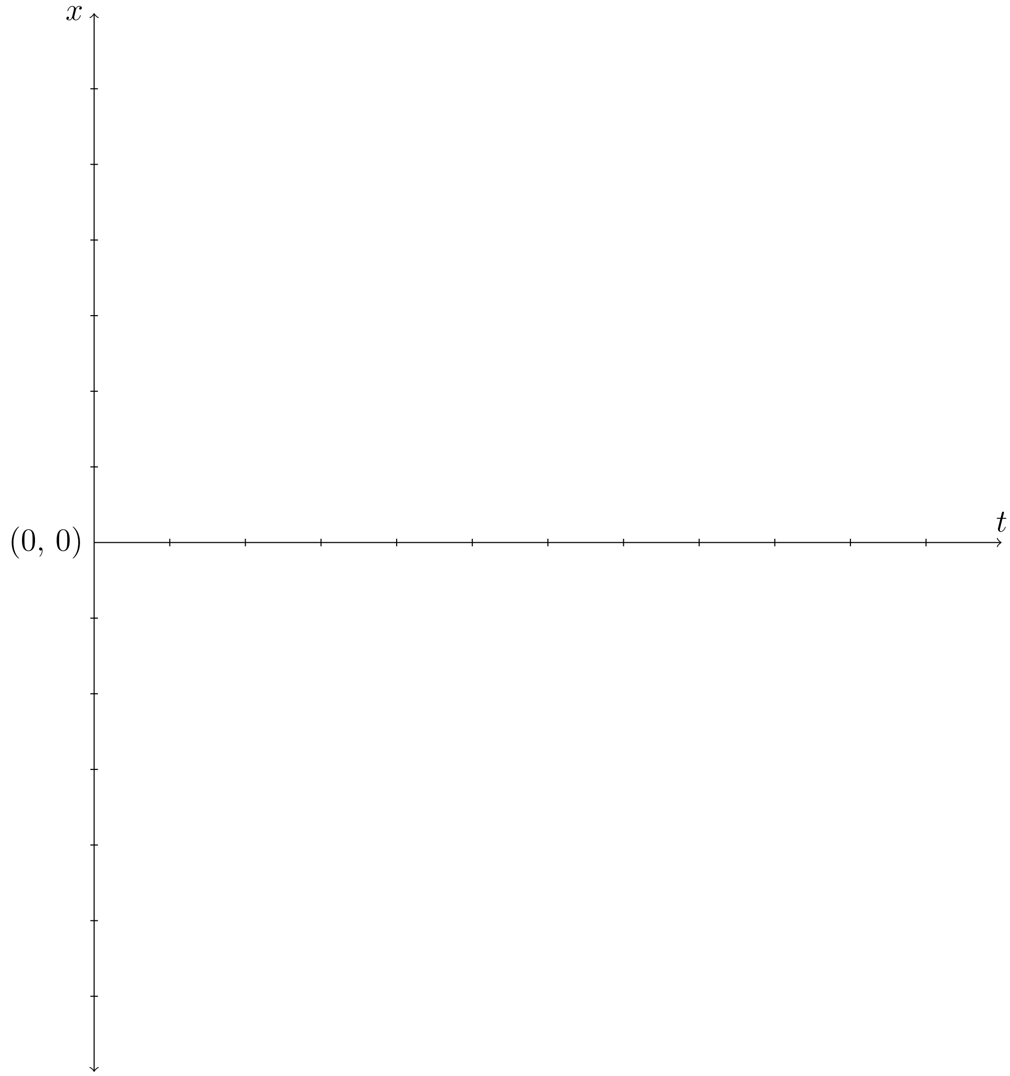
(a) Where will the graph of $p(t)$ intersect the $t$ and $x$ axes?

(b) Does the graph of $p(t)$ have any asymptotes? If so, where are they?

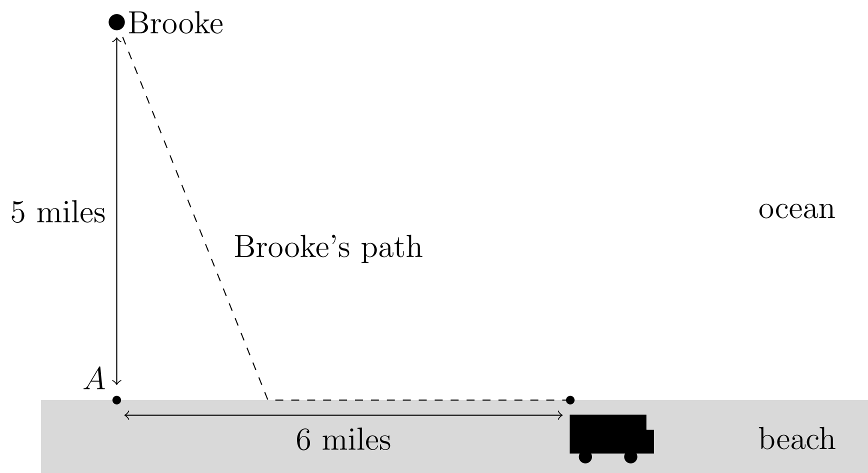
(c) Where is $p(t)$ increasing and where is it decreasing?

(d) What are the $t$-coordiates of the local minima and maxima?

(e) Using all the information above, and your answers to the questions, sketch a graph of $p(t)$ on the plane below.



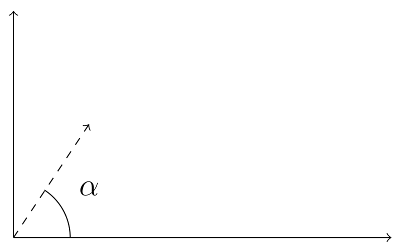
2. Brooke is located 5 miles out at sea from a straight shoreline in her kayak. She wants to make it to the taco truck on the beach for lunch, which is 6 miles from point $A$ on shore (see picture). Brooke can paddle 2 miles/hour and walk 4 miles/hour. If she paddles along a straight line to shore, find an equation for the total time it will take Brooke to get to lunch. Your equation will depend on where Brooke beaches the boat. Where should she land the boat to eat as soon as possible?



# Worksheet for Week 11: Maximizing functions

The best way to study for the Math 124 final is to solve challenging problems. In this worksheet, you'll solve a difficult problem that incorporates several topics from the class: position and velocity, parametric equations, derivatives, and maximizing functions by checking for critical points and also checking the endpoints.

You're on the planet Zed, which has no air and has gravitational constant $-g$. (Earth's gravitational constant is about $-9.8m/s^2$.) You throw a ball at speed $v$ from the origin on a coordinate plane, where the horizontal axis lies along the ground of Zed. (See the picture.) The angle $\alpha$ is the angle above the horizontal that the ball is thrown. Suppose $0 \leq \alpha \leq \pi/2$.




Let $x(t)$, $y(t)$ denote the position of the ball at time $t$, where $t = 0$ is the instant the ball was thrown in the air. Then $x'(t)$, $y'(t)$ denote the horizontal and vertical velocities as functions of $t$.

In this situation (airless planet with gravitational constant $-g$), we have the formulas

$$y(t) = -\frac{1}{2}gt^2 + y'(0)t + y(0),$$
$$x(t) = x'(0)t + x(0).$$

1. What are $x'(0)$, $y'(0)$, $x(0)$ and $y(0)$? Use these numbers and the formulas above to find equations for $x'(t)$ and $y'(t)$.

2. When does the ball hit the ground?

3. When does the ball reach the peak of its trajectory?

4. How far from the origin is the ball when it hits the ground?

5. Which angle $\alpha$ will maximize the distance the ball travels? Your answer will involve the constants $g$ and $v$.

6. In this problem, you'll figure out when the ball is farthest from the origin. The answer will depend on the angle $\alpha$.

   (a) Find a formula $F(t)$ for the **square of** the distance from the ball to the origin at time $t$. (Maximizing the square of the distance is the same as maximizing the distance, and this way is less messy.)

(b) Find an expression for the non-zero critical numbers of $F(t)$.

(c) For some values of $\alpha$, there are no non-zero critical numbers to check. For which $\alpha$ are there **no** non-zero roots of $F'(t)$?

(d) For the values of $\alpha$ you found in part (c), when is the ball the farthest from the origin?

(e) For which angle(s) $\alpha$ is there only one non-zero critical number of $F(t)$ to check?

(f) For the values of $\alpha$ you found in part (e), when is the ball the farthest from the origin? Be sure to check your answer.

(g) Are there any angles $\alpha$ so that the ball is farthest from the origin while it's in the air?