Axioms

Suppose x, y, and z are real numbers. We will take as fact each of the following.

1. $x + y$ and xy are real numbers. ($\mathbb{R}$ is *closed* under addition and multiplication.)
2. If $x = y$, then $x + z = y + z$ and $xz = yz$. (This is sometimes called *substitution of equals*.)
3. $x + y = y + x$ and $xy = yx$ (addition and multiplication are *commutative* in $\mathbb{R}$)
4. $(x+y)+z = x+(y+z)$ and $(xy)z = x(yz)$ (addition and multiplication are *associative* in $\mathbb{R}$)
5. $x(y + z) = xy + xz$ (This is the *Distributive Law*.)
6. $x + 0 = 0 + x = x$ and $x \cdot 1 = 1 \cdot x = x$ (0 is the *additive identity*; 1 is the *multiplicative identity*.)
7. There exists a real number $-x$ such that $x+(-x) = (-x) + x = 0$. (That is, every real number has an *additive inverse* in $\mathbb{R}$.)
8. If $x \neq 0$, then there exists a real number x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. (That is, every non-zero real number has a *multiplicative inverse* in $\mathbb{R}$.)
9. If $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$.
10. Either $x > 0$, $-x > 0$, or $x = 0$.
11. If x and y are integers, then $-x$, $x + y$, and xy are integers. (The additive inverse of an integer is an integer and $\mathbb{Z}$ is closed under addition and multiplication.)

NOTE: It is not hard to prove that $\mathbb{Q}$, the set of rational numbers is closed under addition and multiplication and that every non-zero rational number has a multiplicative inverse in $\mathbb{Q}$.

Elementary Properties of Real Numbers

The following properties of real numbers that allow us to do algebra follow from the axioms on the front page.

If x, y, z, u, and v are real numbers, then:

1. $x \cdot 0 = 0$
2. If $x + z = y + z$, then $x = y$.
3. If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.
4. $-x = (-1) \cdot x$
5. $(-x) \cdot y = -(x \cdot y)$
6. $(-x) \cdot (-y) = x \cdot y$
7. If $x \cdot y = 0$, then $x = 0$ or $y = 0$.
8. If $x \leq y$ and $y \leq x$, then $x = y$.
9. If $x \leq y$ and $y \leq z$, then $x \leq z$.
10. At least one of the following is true: $x \leq y$ or $y \leq x$.
11. If $x \leq y$, then $x + z \leq y + z$.
12. If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.
13. If $x \leq y$ and $z \leq 0$, then $yz \leq xz$.
14. If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$.
15. If $0 \leq x \leq y$ and $0 \leq u \leq v$, then $xu \leq yv$.
16. If $x \leq y$, then $-y \leq -x$.
17. $0 \leq x^2$
18. $0 < 1$
19. If $0 < x$, then $0 < x^{-1}$.
20. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

And here are a couple of properties of integers.

21. Every integer is either even or odd, never both.
22. The only integers that divide 1 are -1 and 1.