Theorem Suppose $a, n \in \mathbb{Z}_{>0}$ and $\sqrt[n]{a}$ is rational. Then $\sqrt[n]{a}$ is an integer.

Proof Suppose $a, n \in \mathbb{Z}_{>0}$.
Suppose that $\sqrt[n]{a}$ is rational.
Note that if $a = 1$, then $\sqrt[n]{a} = \sqrt[n]{1} = 1$, which is integer. So we may now suppose that $a > 1$.
Then there exist integers C and D such that $\sqrt[n]{a} = \frac{C}{D}$, and $(C, D) = 1$.
Then

$$aD^n = C^n.$$

Let p be any prime that divides a.
Then $p \mid C^n$ and so $p \mid C$.
Let k be the integer such that $p^k \mid\mid C$ (i.e., let $k = \mathrm{ord}_p C$).
Then $p^{nk} \mid\mid C^n$.
Since $(C, D) = 1$, we can conclude that $p^{nk} \mid\mid a$.
This shows that the exponent on any prime in the prime factorization of a is a multiple of n.
That is,

$$a = p_1^{nk_1} p_2^{nk_2} \cdots p_r^{nk_r}$$

for some set of primes $\{p_1, p_2, \ldots, p_r\}$ and some set of positive integers $\{k_1, k_2, \ldots, k_r\}$.
Hence,

$$\sqrt[n]{a} = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

so $\sqrt[n]{a}$ is a product of integer powers of primes, and hence is an integer. ■