Axioms of the Integers (AIs)

Suppose a, b, and c are integers.

- **Closure:**

  $a+b$ and ab are integers.

- **Substitution of Equals:**

  If $a=b$, then $a+c=b+c$ and $ac=bc$.

- **Commutativity:**

  $a+b=b+a$ and $ab=ba$.

- **Associativity:**

  $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.

- **The Distributive Law:**

  $a(b+c)=ab+ac$

- **Identities:**

  $a+0=0+a=a$ and $a\cdot 1=1\cdot a=a$

  0 is called the *additive identity*

  1 is called the *multiplicative identity*.

- **Additive Inverses:**

  There exists an integer $-a$ such that $a+(-a)=(-a)+a=0$.

- **Trichotomy:**

  Exactly one of the following is true: $a<0$, $-a<0$, or $a=0$.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. $a\cdot 0=0$
2. If $a+c=b+c$, then $a=b$.
3. $-a=(-1)\cdot a$
4. $(-a)\cdot b=-(a\cdot b)$
5. $(-a)\cdot(-b)=a\cdot b$
6. If $a\cdot b=0$, then $a=0$ or $b=0$.
7. If $a\leq b$ and $b\leq a$, then $a=b$.
8. If $a<b$ and $b<c$, then $a<c$.
9. If $a<b$, then $a+c<b+c$.
10. If $a<b$ and $0<c$, then $ac<bc$.
11. If $a<b$ and $c<0$, then $bc<ac$.
12. If $a<b$ and $c<d$, then $a+c<b+d$.
13. If $0\leq a<b$ and $0\leq c<d$, then $ac<bd$.
14. If $a<b$, then $-b<-a$.
15. $0\leq a^2$
16. If $ab=1$, then either $a=b=1$ or $a=b=-1$.

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.