Axioms and Elementary Properties of the Integers

The set of integers is denoted $\mathbb{Z}$.

Equality is reflexive, symmetric and transitive. This means:

(a) $x = x$ for any $x \in \mathbb{Z}$ (this is the property of *reflexivity*).

(b) For any $x, y \in \mathbb{Z}$, if $x = y$, then $y = x$ (this is the property of *symmetry*).

(c) For any $x, y, z \in \mathbb{Z}$, if $x = y$ and $y = z$, then $x = z$ (this is the property of *transitivity*).

We define subtraction like this: $a - b = a + (-b)$ for any $a, b \in \mathbb{Z}$.

We define the statement $a < b$ to be equivalent to $a + (-b) < 0$.

We define the statement $a > b$ to be equivalent to the statement $b < a$.

We define exponents as follows: For n a positive integer, $a^n = a \cdot a \cdots a$ where there are n instances of a on the right hand side.

Specific numbers are defined in the standard way: 2=1+1,3=1+1+1, etc.

Axioms of the Integers (AIs)

Suppose a, b, and c are integers.

- **Closure:**

  $a + b$ and ab are integers.

- **Substitution of Equals:**

  If $a = b$, then $a + c = b + c$ and $ac = bc$.

- **Commutativity:**

  $a + b = b + a$ and $ab = ba$.

- **Associativity:**

  $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

- **The Distributive Law:**

  $a(b + c) = ab + ac$

- **Identities:**

  $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$

  0 is called the *additive identity*

  1 is called the *multiplicative identity*.

- **Additive Inverses:**

  There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.

- **Trichotomy:**

  Exactly one of the following is true: $a < 0$, $-a < 0$, or $a = 0$.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(a \cdot b)$
5. $(-a) \cdot (-b) = a \cdot b$
6. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.