1. Prove that, for all $x \in \mathbb{Z}$, if $x^2 - 1$ is divisible by 8, then x is odd.

   Proof: Suppose x is an even integer.

   Then $x = 2k$ for some integer k.

   Then $x^2 - 1 = (2k)^2 - 1 = 4k^2 - 1$.

   Suppose $x^2 - 1$ is divisible by 8.

   Then there exists an integer m such that $4k^2 - 1 = 8m$.

   Then $1 = 8m - 4k^2 = 4(2m - k^2)$.

   Since $2m - k^2 \in \mathbb{Z}$, 4 divides 1.

   However, 4 does not divide 1 since $0 < 1 < 4$, so we have a contradiction.

   Hence, the assumption that $x^2 - 1$ is divisible by 8 is false.

   Thus $x^2 - 1$ is not divisible by 8.

   Hence, if $x^2 - 1$ is divisible by 8, then x is odd. ■

2. Let a and b be integers. Prove that $x = a^2 + ab + b$ is odd iff a is odd or b is odd.

   Proof: Suppose a and b are integers.

   We can consider four cases.

   (a) Suppose a is even and b is even.
       Then $a = 2k$ and $b = 2m$ for some integers k and m.
       Then $x = 4k^2 + 4km + 2m = 2(2k^2 + 2km + m)$.
       Since $2k^2 + 2km + m$ is an integer (by Closure), x is even.

   (b) Suppose a is even and b is odd.
       Then $a = 2k$ and $b = 2m + 1$ for some integers k and m.
       Then $x = 4k^2 + 2k(2m + 1) + 2m + 1 = 2(2k^2 + k(2m + 1) + m) + 1$.
       Since $2k^2 + k(2m + 1) + m$ is an integer (by Closure), x is odd.

   (c) Suppose a is odd and b is even.
       Then $a = 2k + 1$ and $b = 2m$ for some integers k and m.
       Then

       $$\begin{aligned}
       x &= (2k + 1)^2 + (2k + 1)(2m) + 2m \\
         &= 4k^2 + 4k + 1 + (2k + 1)(2m) + 2m \\
         &= 2(2k^2 + 2k + m(2k + 1) + m) + 1.
       \end{aligned}$$

       Since $2k^2 + 2k + m(2k + 1) + m$ is an integer (by Closure), x is odd.

   (d) Suppose a is odd and b is odd.
       Then $a = 2k + 1$ and $b = 2m+!$ for some integers k and m.
       Then

       $$\begin{aligned}
       x &= (2k + 1)^2 + (2m + 1)(2k + 1) + b \\
         &= 4k^2 + 4k + 1 + 4mk + 2m + 2k + 1 + 2m + 1 \\
         &= 2(2k^2 + 2mk + 3k + 2m + 1) + 1.
       \end{aligned}$$

       Since $(2k^2 + 2mk + 3k + 2m + 1)$ is an integer (by Closure), x is odd.

The first case shows that if a and b are even, then x is even; hence, if x is odd, then a is odd or b is odd.

The other three cases show that if a or b is odd, then x is odd.

Hence, x is odd if and only if a is odd or b is odd. ∎

3. Let a and b be integers. Prove that $a(b + a + 1)$ is odd iff a and b are both odd.

   Proof: Let a and b be integers.

   Suppose a is even.

   Then $a = 2k$ for some $k \in \mathbb{Z}$.

   Then $a(b + a + 1) = 2k(b + a + 1)$, and since a, b, and k are integers, $k(b + a + 1)$ is an integer (by Closure).

   Hence, $2 \mid a(b + a + 1)$, i.e., $a(b + a + 1)$ is even.

   Suppose a is odd and b is even.

   Then $a = 2k + 1$ and $b = 2m$ for some integers k and m.

   Then $a(b + a + 1) = a(2k + 2m + 1 + 1) = a(2k + 2m + 2) = 2a(k + m + 1)$.

   Since a, k, and m are integers, $a(k + m + 1)$ is an integer (by Closure), and so $2 \mid a(b + a + 1)$, i.e., $a(b + a + 1)$ is even.

   Suppose a is odd and b is odd.

   Then $a = 2k + 1$ and $b = 2m + 1$ for some integers k and m.

   Then

   $$
   \begin{aligned}
   a(b + a + 1) &= (2k + 1)(2k + 2m + 3) \\
   &= 4k^2 + 4mk + 8k + 2m + 3 \\
   &= 2(2k^2 + 4mk + 4k + m + 1) + 1.
   \end{aligned}
   $$

   Since k and m are integers, $2k^2 + 4mk + 4k + m + 1$ is an integer (by Closure), and so $a(b + a + 1)$ is odd.

   Thus, $a(b + a + 1)$ is odd iff a and b are both odd. ∎

4. Prove or give a counterexample for each of the following statements.

   (a) For all integers x and y, $|xy| = |x||y|$.
      Proof: Here, I give only the outline of the proof.
      Suppose x and y are integers.
      By the trichotomy law, we know $x \geq 0$ or $x < 0$, and $y \geq 0$ or $y < 0$.
      So, we consider four cases:

      i. $x \geq 0, y \geq 0$
      ii. $x \geq 0, y < 0$
      iii. $x < 0, y \geq 0$
      iv. $x < 0, y < 0$.

In each case, use the definition of absolute value to simplify $|x|$ and $|y|$. Also, use elementary integer properties to conclude whether xy is negative or not, and from that, simplify $|xy|$.

(b) For all integers a and b, if $a|b$ and $b|a$, then $a = b$ or $a = -b$.

Proof: Let a and b be non-zero integers.

Suppose $a|b$ and $b|a$.

Then there exist integers k and m such that $b = ak$ and $a = bm$.

Then, by the Substitution of Equals axiom, we have $b = bmk$. Hence, $b - bmk = 0$, and so $b(1 - mk) = 0$.

As $b \neq 0$, we may conclude that $1 - mk = 0$ (by EPI 6).

That is, $mk = 1$ (by Substitution of Equals).

Hence, $k = 1$ or $k = -1$ (by EPI 16), so $b = a$ or $b = -a$. ■

(c) For all integers m and n, if $n + m$ is odd, then $n \neq m$.

Proof: Let m and n be integers.

Suppose $n = m$. Then $n + m = 2m$ which is even.

Hence, $n = m$ implies $n + m$ is even, and so if $n + m$ is odd, then $n \neq m$. ■

5. Let A, B, and C be sets. Prove that $A \cap B = A \setminus (A \setminus B)$.

Proof: Let A, B, and C be sets.

Suppose $x \in A \cap B$.

Then $x \in A$ and $x \in B$.

Hence, $x \notin A \setminus B$.

Since $x \in A$, we conclude $x \in A \setminus (A \setminus B)$.

Thus, $x \in A \cap B$ implies $x \in A \setminus (A \setminus B)$, so $A \cap B \subseteq A \setminus (A \setminus B)$.

Now, suppose $x \in A \setminus (A \setminus B)$.

Then $x \in A$.

Suppose $x \notin B$.

Then $x \in A \setminus B$, and hence $x \notin A \setminus (A \setminus B)$.

This is a contradiction, since $x \in A \setminus (A \setminus B)$.

Thus $x \in B$, and so $x \in A \cap B$.

Hence, $x \in A \setminus (A \setminus B)$ implies $x \in A \cap B$.

Therefore $A \setminus (A \setminus B) \subseteq A \cap B$.

Thus, $A \cap B \subseteq A \setminus (A \setminus B)$ and $A \setminus (A \setminus B) \subseteq A \cap B$, and so $A \cap B = A \setminus (A \setminus B)$. ■

6. Let A, B and C be sets. Prove that $(A \cup B) \setminus (A \cup C) = B \setminus (A \cup C)$.

Proof: Let A, B and C be sets.

Suppose $x \in (A \cup B) \setminus (A \cup C)$.

Then $x \in A \cup B$ and $x \notin A \cup C$.

Hence, $x \notin A$.

Since $x \in A \cup B$, we conclude that $x \in B$.

Hence, $x \in B \setminus (A \cup C)$.

Thus, $x \in (A \cup B) \setminus (A \cup C)$ implies $x \in B \setminus (A \cup C)$, and so $(A \cup B) \setminus (A \cup C) \subseteq B \setminus (A \cup C)$.

Now, suppose $x \in B \setminus (A \cup C)$.

Then $x \in B$, so $x \in A \cup B$.

Also, $x \notin A \cup C$, and so $x \in (A \cup B) \setminus (A \cup C)$.

Hence, $x \in B \setminus (A \cup C)$ implies $x \in (A \cup B) \setminus (A \cup C)$, so $B \setminus (A \cup C) \subseteq x \in (A \cup B) \setminus (A \cup C)$.

Thus, $(A \cup B) \setminus (A \cup C) \subseteq B \setminus (A \cup C)$ and $B \setminus (A \cup C) \subseteq x \in (A \cup B) \setminus (A \cup C)$.

Hence, $(A \cup B) \setminus (A \cup C) = B \setminus (A \cup C)$. ■

7. Let A, B and C be sets. Prove that $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

   Proof: Let A, B and C be sets.

   Suppose $x \in (A \setminus B) \setminus C$.

   Then $x \in A$, $x \notin B$ and $x \notin C$.

   Suppose $x \in B \cup C$.

   Then $x \in B$ or $x \in C$.

   This is a contradiction since $x \notin B$ and $x \notin C$.

   Thus, $x \notin B \cup C$.

   Hence, $x \in A \setminus (B \cup C)$.

   So, $x \in (A \setminus B) \setminus C$ implies $x \in A \setminus (B \cup C)$, and hence $(A \setminus B) \setminus C \subseteq A \setminus (B \cup C)$.

   Now, suppose $x \in A \setminus (B \cup C)$.

   Then $x \in A$ and $x \notin B \cup C$.

   Then $x \notin B$ and $x \notin C$, and so $x \in A \setminus B$, and $x \in (A \setminus B) \setminus C$.

   Thus, $x \in A \setminus (B \cup C)$ implies $x \in (A \setminus B) \setminus C$, and so $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$.

   Hence, $(A \setminus B) \setminus C \subseteq x \in A \setminus (B \cup C)$ and $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$.

   Therefore, $(A \setminus B) \setminus C = A \setminus (B \cup C)$. ■

8. Let A, B, and C be sets. Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

   Proof: Let A, B, and C be sets.

   Suppose $A \cup C \subseteq B \cup C$.

   Suppose $x \in A \setminus C$.

   Then $x \in A$ and $x \notin C$.

   Then $x \in A \cup C$, so $x \in B \cup C$.

   Since $x \notin C$, we conclude that $x \in B$.

   Hence, $x \in B$ and $x \notin C$, i.e., $x \in B \setminus C$.

   Thus, $x \in A \setminus C$ implies $x \in B \setminus C$, so $A \setminus C \subseteq B \setminus C$.

   Therefore, $A \cup C \subseteq B \cup C$ implies $A \setminus C \subseteq B \setminus C$.

Now, suppose $A \setminus C \subseteq B \setminus C$.

    Suppose $x \in A \cup C$.

        Then $x \in A$ or $x \in C$.

            Suppose $x \in C$.

                Then $x \in B \cup C$.

            Suppose $x \notin C$.

                Then $x \in A$, and so $x \in A \setminus C$, and hence $x \in B \setminus C$.

                So $x \in B$, and hence $x \in B \cup C$.

    Hence, $x \in B \cup C$, and so $x \in A \cup C$ implies $x \in B \cup C$.

That is, $A \cup C \subseteq B \cup C$.

Thus, $A \setminus C \subseteq B \setminus C$ implies $A \cup C \subseteq B \cup C$.

Thus, $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$. ∎