MATH 300 C - Spring 2016
Midterm 2 Practice Problems

1. Prove that, for all integers n, 3 does not divide $n^2 - 5$.

2. Define a relation R on $\mathbb{Z}$ by
$$(x, y) \in R \Leftrightarrow 4 \mid x^2 - y^2.$$
Is R an equivalence relation? Prove your answer.

3. Use induction to prove that
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$
for all $n \in \mathbb{Z}_{>0}$.

4. Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$.
   (a) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.
   (b) Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.

5. Let $A = \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$.
Thus, A is the xy-plane without the origin.
Define a relation R on A by
$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1) \text{ and } (x_2, y_2) \text{ lie on a line which passes through the origin.}$$
Prove that R is an equivalence relation.

6. Let $S = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. Define a relation $R \subseteq S \times S$ by
$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1 y_1 = x_2 y_2.$$
Prove that R is an equivalence relation.

7. Let $\mathcal{F}$ be a family of sets, and B be a set. Prove that if $\bigcup \mathcal{F} \subseteq B$, then $\mathcal{F} \subseteq \mathcal{P}(B)$.

8. Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$.

9. Give an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that f is one-to-one, but not onto (i.e., f is injective but not surjective). Prove that f is one-to-one and not onto.

10. Give an example of a function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that g is onto, but not one-to-one (i.e., g is surjective, but not injective). Prove that g is onto and not one-to-one.

11. Use induction to prove that $n! > n^2$ for all integers $n \geq 4$.

12. Let R be the relation defined on the real numbers, $\mathbb{R}$, by
$$(x, y) \in R \Leftrightarrow \text{there exist positive integers } n \text{ and } m \text{ such that } x^n = y^m.$$
Prove that R is an equivalence relation.

13. Let A, B and C be sets. Let $f : A \to B$, and $g : B \to C$.

    (a) Suppose $g \circ f : A \to C$ is one-to-one. Is f necessarily one-to-one? Prove your answer.

    (b) Suppose $g \circ f : A \to C$ is one-to-one. Is g necessarily one-to-one? Prove your answer.

14. Let S be a set.
    Define a function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ by $f(A) = S \setminus A$ for all $A \in \mathcal{P}(S)$.
    Prove that f is a bijection.

15. Let S be the set of all functions $f : \mathbb{R} \Rightarrow \mathbb{R}$. Define a relation R on S by

$$(f, g) \in R \Leftrightarrow \exists c \in \mathbb{R}, c \neq 0, \text{ such that } f(x) = cg(x) \text{ for all } x \in \mathbb{R}.$$

    Prove that R is an equivalence relation.

16. Let A and B be sets.
    Let f and g be functions from A to B.
    Prove that if $f \cap g \neq \varnothing$, then $f \setminus g$ is not a function from A to B.

17. Let $n \in \mathbb{Z}_{>0}$.
    Use induction to prove $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.