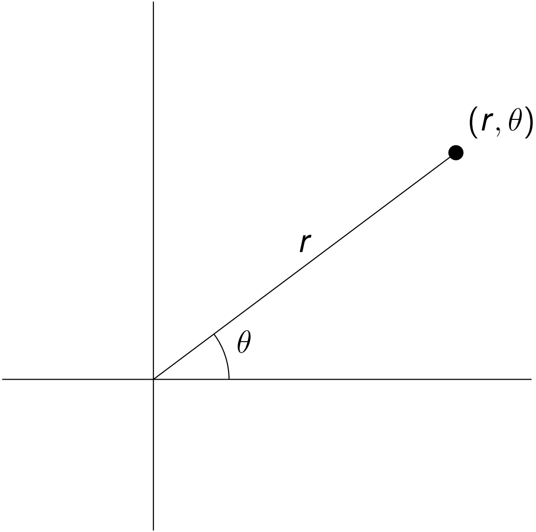
# Curves defined using polar coordinates

Instead of using Cartesian $(x, y)$ coordinates, we can describe points in the plane by specifying how far they are from some fixed reference point (the **pole**) and the angle the line from the point to the pole makes with some reference direction.

The convention is to treat the origin $(0, 0)$ as the pole, and the reference direction as the positive $x$-axis. In this way, we can convert back and forth between polar coordinates and cartesian.

So a point in the plane may be specified by a pair $(r, \theta)$ like this:



Here are some points in both *xy*-coordinates and polar coordinates:

| Cartesian | polar |
|---|---|
| $(1, 0)$ | $(1, 0)$ |
| $(1, 1)$ | $(\sqrt{2}, \frac{\pi}{4}$ |
| $(0, 1)$ | $(1, \sqrt{\pi}2$ |
| $(-1, 0)$ | $(1, \pi)$ |
| $(-1, -1$ | $(\sqrt{2}, \frac{5\pi}{4}$ |
| $(0, -1)$ | $(1, \frac{3\pi}{2}$ |
| $(3, 7)$ | $(7.6157..., 1.1659...)$ |
| $(0.5673..., -1.9178)$ | $(2, 5)$ |

To convert back and forth, we can use these relationships:

$$x = r\cos\theta, \;\; y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \;\; \theta = \tan^{-1}\frac{y}{x}$$

with the caveat that the theta value yielded might be negative.

Which brings us the main oddity of polar coordinates: they are not unique.

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For example,

$$(r, \theta), (-r, \theta + \pi), (-r, \theta + 3\pi), (r, \theta + 2\pi)$$

all represent the same point.

This makes thing interesting: simple polar equations can create complex curves.

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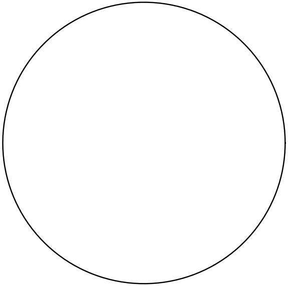
We will just look at curves defined by equations of the form $r = f(\theta)$ where $f$ is some reasonably nice function.

We'll start at the simplest and work our way up a ways.

The simplest possibility is to have $f$ be a constant function.

In this case, we get a circle: the equation $r = c$ is the circle centered at $(0, 0)$ with radius $c$.

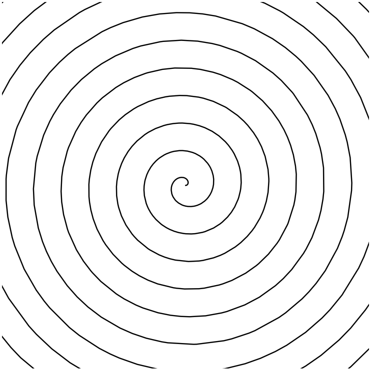


The next level of complexity is to have $f$ be a linear function, so that

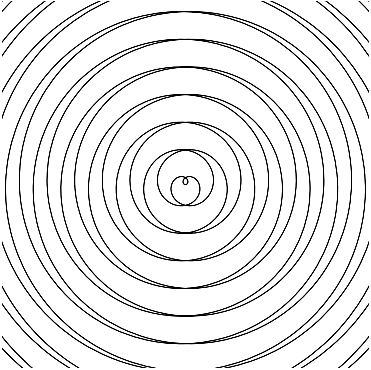
$$r = a\theta + b.$$

The simplest possible form of this is $r = \theta$.

This yields the Archimedean spiral (here with $\theta > 0$):



If we allow any value of $\theta$ we get the more mysterious Archimedean spiral:



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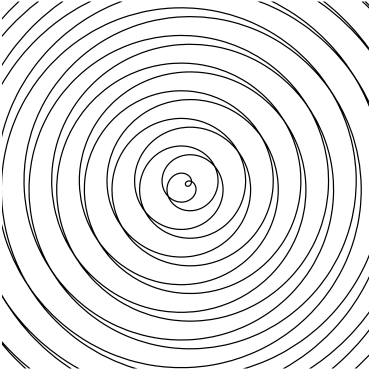
The *b* value rotates the spiral around the origin without changing it in any other way, so you can just tilt your head to get the same effect.

Just for an example, here is $r = \theta + 1$:

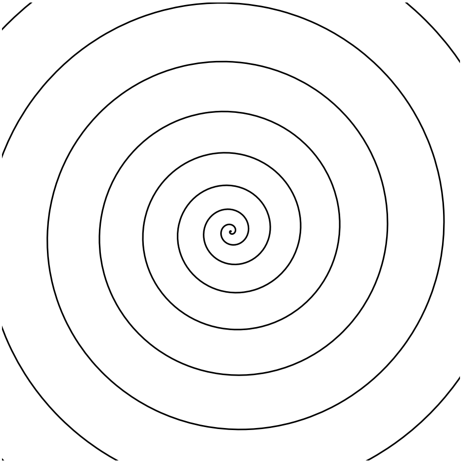


The reason $r = \theta$ looks like a spiral is basically because $f(\theta) = \theta$ is **monotonic**.

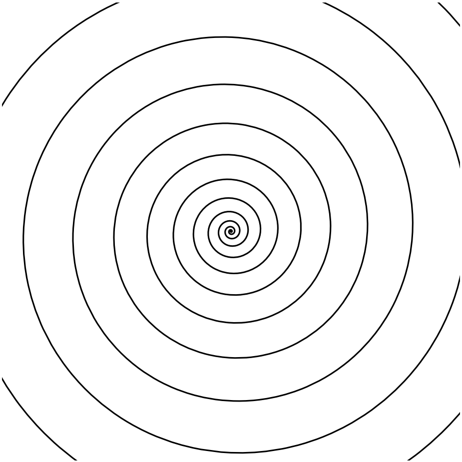
Monotonic functions are those that are always increasing or decreasing.

With this in mind, we can generate other spirals by choosing $f$ to be various monotonic functions.

Here is $r = \frac{1}{50}\,\theta^2$:



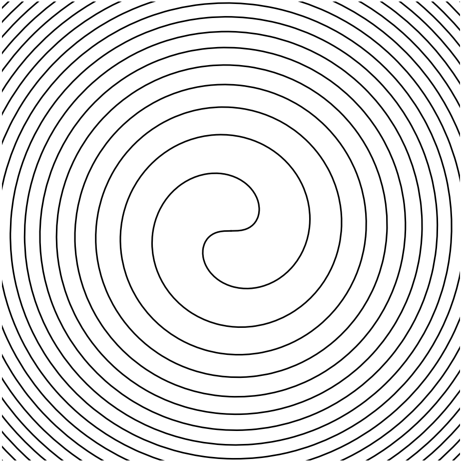
Here is $r = \frac{1}{10000}\,\theta^3$:



In general $r = a\theta^k$ for $k > 1$ yields a pretty similar-looking spiral, with increasing space between each turn.

For $0 < k < 1$, we get spirals with **decreasing** space between each turn:

Here is $r^2 = \theta$; that is, $r = \pm\,\theta^{0.5}$:



The most important spiral is the **equiangular spiral**, whose equation is

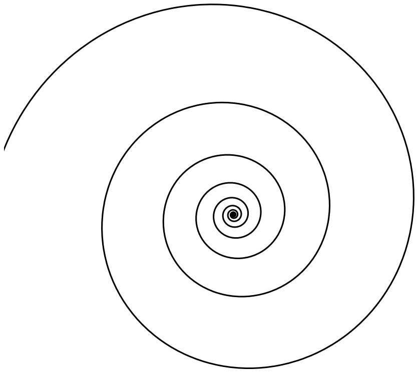
$$r = ae^{b\theta}.$$

It is also known as the logarithmic spiral, or *spira mirabilis*.

One thing that makes this curve different from the previous ones is that we can let $\theta$ run from $-\infty$ to $\infty$: as $\theta$ goes to $\infty$, $r$ goes to $\infty$, while $r$ goes to zero as $\theta$ goes to $-\infty$.

So the spiral spirals in and out forever.

Here is $r = \frac{1}{10} e^{\frac{\theta}{10}}$:

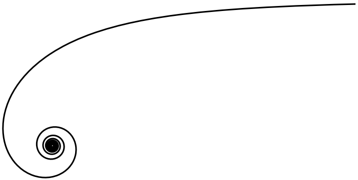


If you zoom in on the origin, it would look pretty much like this forever.

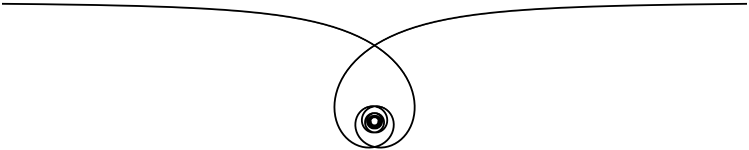
A different sort of spiral arises with the equation

$$r = \frac{1}{\theta}$$

Here it is for $\theta > 0$:

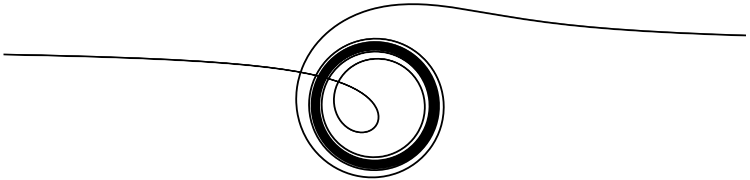


Here it is for $\infty < \theta < \infty$:

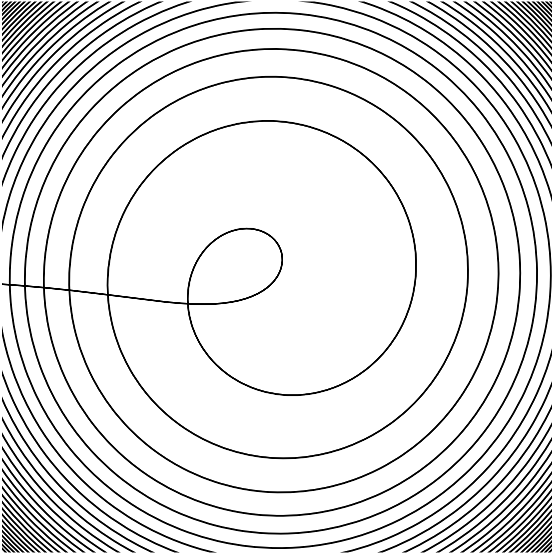


Yes: it has *y* $= 1$ as a horizontal asymptote: as *theta* goes to zero, *y* goes to 1 and *x* goes to $\infty$.

Here is $r = 1 + \frac{1}{\theta}$:



Here is $r = \ln \theta$:

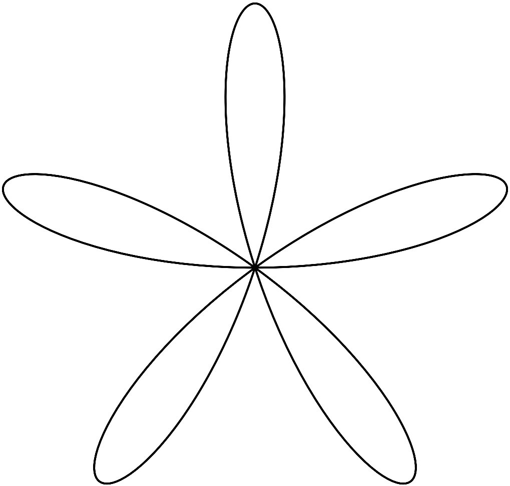


Okay, what if $f$ is not monotonic?

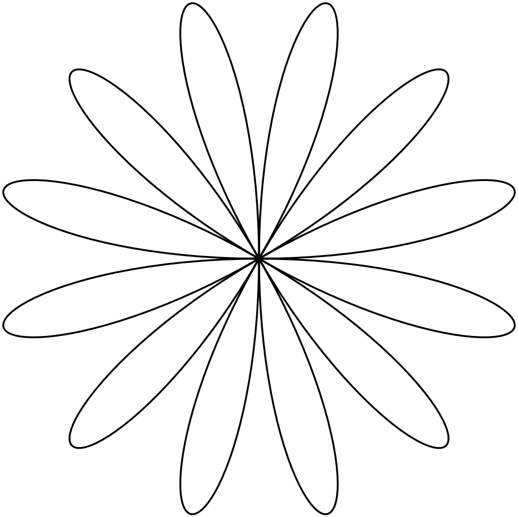
The classic example is curves of the form $r = \sin a\theta$ where $a$ is a rational number.

When $a$ is an integer, there is a very particular result: we get a curve called a **rose**.

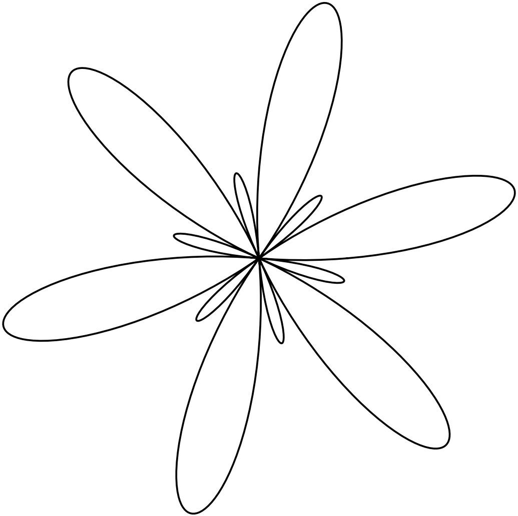
If *a* is odd, we get *a* petals. Here is $r = \sin 5\theta$:



If *a* is even, we get $2a$ petals. Here is $r = \sin 6\theta$.

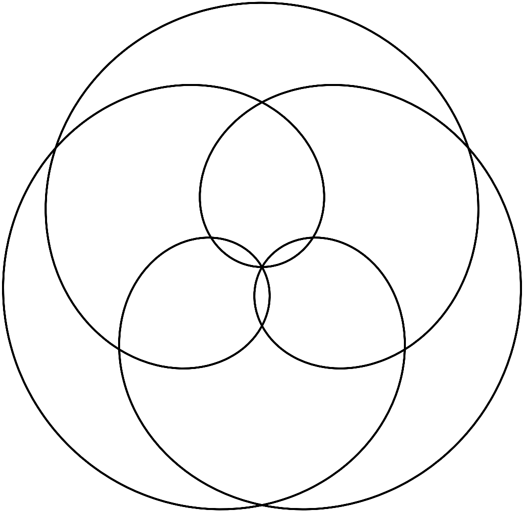


Here is $r = \frac{1}{2} + \sin 6\theta$:

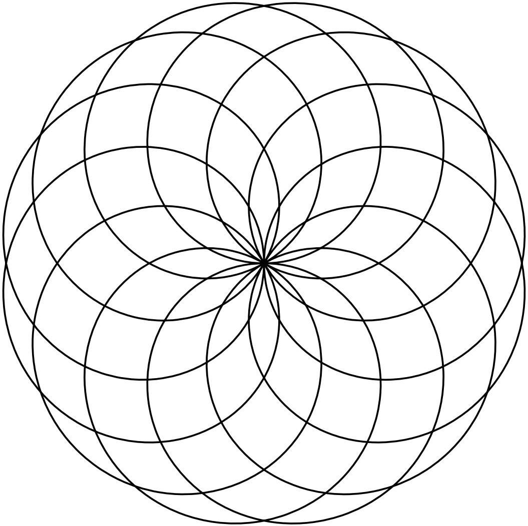


If *a* is rational, but not an integer, there is some variety.

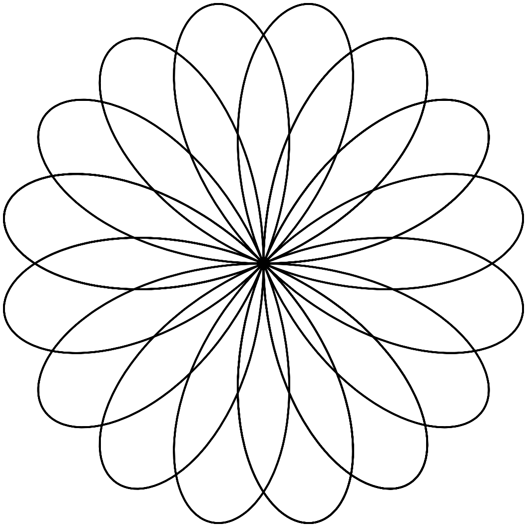
Here is $r = \sin \frac{3}{7}\theta$:



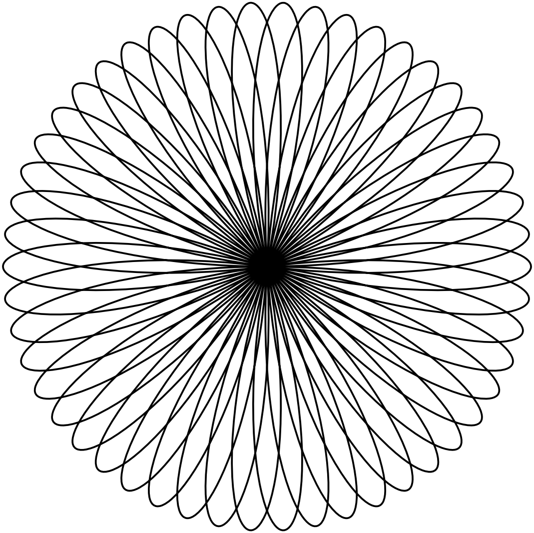
Here is $r = \sin \frac{6}{7}\theta$:



Here is $r = \sin \frac{8}{3}\theta$:



Here is $r = \sin \frac{25}{4}\theta$:



A little modification: $r = \frac{1}{2} + \sin^2 \frac{14}{3}\theta$: