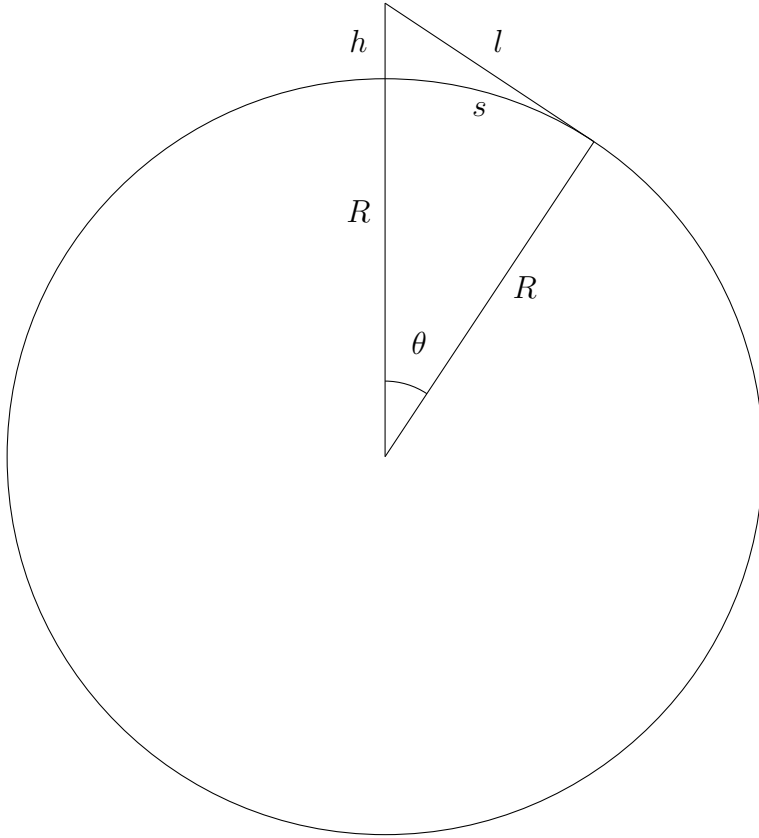


## How far away is the horizon?

When we stand on the shore, with our feet just at sea level, and look out to sea, we may ask: how far away is the horizon?

We start by making the simplifying assumption that the earth is a sphere with radius  $R$ . Then the line from our eye to the horizon is tangent to a circle of radius  $R$  (representing the earth) as shown:



Here  $h$  is our height (or the height of our eye) above sea level,  $R$  is the radius of the earth,  $l$  is the length of the line connecting our eye directly to the horizon, and  $s$  is the arc length, the distance along the surface of the earth to the horizon. The angle  $\theta$  is the angle made between our position and the horizon.

Since the triangle in the figure is a right triangle, we may use the Pythagorean theorem to find relationships between the variables.

Specifically,

$$l = \sqrt{(R + h)^2 - R^2} = \sqrt{2Rh + h^2}.$$

If we make the assumption that  $l$  should be approximately the same as  $s$  (since  $h$  will be very small compared to  $R$ ), then we can say

$$s \approx l = \sqrt{2Rh + h^2} \approx \sqrt{2Rh} = k\sqrt{h}$$

for a constant  $k$ . Here we have made the additional simplification that  $2Rh + h^2 \approx 2Rh$  since  $h^2$  will be very small compared to  $2Rh$ .

Working out this  $k$  value, we can use  $R = 3959$  miles so that, with  $h$  in feet, we have

$$s \approx \sqrt{2Rh} = \sqrt{41807040h} = \frac{\sqrt{41807040}}{5280} \sqrt{h} \text{ miles} \approx 1.22459 \sqrt{h} \text{ miles}.$$

For example, if our eye is 5 feet above sea level, then the horizon distance is about

$$1.22459\sqrt{5} = 2.738 \text{ miles away.}$$

To check our calculations, we can work out an “exact” formula and compare.

The angle  $\theta$  can be found as

$$\theta = \cos^{-1} \frac{R}{R+h}$$

so that

$$s = R\theta = R \cos^{-1} \frac{R}{R+h}.$$

This expression is, perhaps surprisingly, extremely close to our previous, simpler expression. Here’s a table of a few values ( $h$  in feet, other distances in miles).

$h$	$1.22459\sqrt{h}$	$s = R \cos^{-1} \frac{R}{R+h}$	percentage difference
5	2.7383	2.7378	-0.000008172203854%
10	3.8725	3.8728	0.000001794221200%
20	5.4765	5.4763	0.00002172705563%
50	8.6592	8.6590	0.00008152553997%
100	12.246	12.246	0.0001811895939%
500	27.383	27.382	0.0009784981053%
1000	38.725	38.724	0.001975123947%
10000	122.459	122.434	0.01991252800%
100000	387.249	386.478	0.1990930664%

As you can see, the simpler formula yields incredible accuracy to the complex one, at least to a height of 100000 feet! Notice that at 100000 feet, the angle  $\theta$  is still only 5.5932 degrees. So, for all practical purposes, we can estimate the distance to the horizon using  $1.22459\sqrt{h}$  (or perhaps even more simply as just  $1.2\sqrt{h}$ ), which gives the distance in miles when  $h$  is in feet. The metric version is

$$s \approx 3.56972\sqrt{h} \approx 3.6\sqrt{h}$$

in kilometers, with  $h$  in meters.