Why are all linear-to-linear rational functions essentially the same?

The family of *linear-to-linear rational functions*, that is, functions of the form

$$f(x) = \frac{ax+b}{cx+d}$$

with $a, b, c,$ and d constant and $ac \neq 0$, are often encountered in mathematics. Although there are four parameters, and the apparent potential for quite a diverse collection of functions in this family, it turns out that all of these functions behave the same way.

To see this, start by dividing $cx+d$ into $ax+b$. The result is

$$f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$$

which we can rewrite as

$$f(x) = \frac{a}{c} + \left(b - \frac{ad}{c}\right)\frac{1}{cx+d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right)\frac{1}{c(x+\frac{d}{c})} = \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{x+\frac{d}{c}}.$$

If we now let

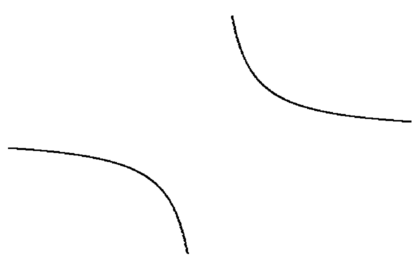
$$A = \frac{a}{c}\,,\ B = \left(\frac{bc-ad}{c^2}\right),\text{ and } C = \frac{d}{c}\,,$$

then we can write

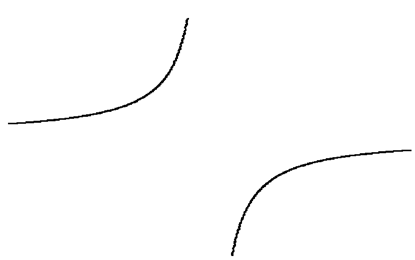
$$f(x) = A + B\frac{1}{x+C}.$$

At this point, we can now see that the graph of $f(x)$ can be thought of as the graph of $\frac{1}{x}$ after it has undergone a series of horizontal and vertical stretches and shifts. As a result, the graphs of all linear-to-linear rational functions look essentially the same as the graph of $\frac{1}{x}$, and hence all look essentially the same.

Note that $f(x)$ has a single vertical asymptote at $x = -\frac{d}{c}$, and a horizontal asymptote of $y = \frac{a}{c}$. With this information, there are essentially two shapes the graph of $f(x)$ could have:



or



To figure out which one it is, all you need is a single point. Sketch the asymptotes, then pick a particular x value to get a point $(x, f(x))$ on the graph, and sketch in the rest.