Math 124 K - Autumn 2008
Mid-Term Exam Number One
October 21, 2008
Answers/Solutions

1. Determine the values of the following limits or state that the limit does not exist. If it is correct to say that the limit equals ∞ or $-\infty$, then you should do so.

   (a) $$\lim_{x\to 0^+} \frac{1+x}{e^{x^2-x}-1} = -\infty$$

   (b) $$\lim_{x\to 3} \frac{\sqrt{x^2-9}}{\sqrt{2x-6}} = \sqrt{3}$$

   (c) $$\lim_{x\to 0^-} \frac{|x-|x||}{|2x-|x||} = 2/3$$

   (d) $$\lim_{x\to\infty} \left(\frac{2x+\sin(x)}{x-3\sin(x)} + \frac{\sin(x^2+x)-\sin(x^2-x)}{x+1} \right) = 2$$

2. Find a point on the curve $y = x^3$ for which the tangent line at that point passes through the point $(0,1)$.

   The point is $\left(-\frac{1}{\sqrt[3]{2}}, -\frac{1}{2}\right)$.

3. Find the x-coordinates of all points on the curve $y = (x+2)(x^2-8x+1)$ at which the tangent line is horizontal.

   The x-coordinates are $x = -1$ and $x = 5$.

4. Let f be defined by the expression $f(x) = x|x|$. If f is differentiable at $x = 0$, find $f'(0)$. If f is not differential at $x = 0$, explain why. In either case, be sure to show all your work.

   $$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)|0+h| - 0|0|}{h} = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0$$

   by the continuity of the absulute value function.

5. Show that if f is an odd function, then the function g defined by $g(x) = |f(x)|$ is an even function.

   We want to shoe that $g(-x) = x$. We have

   $$g(-x) = |f(-x)| = |-f(x)| = |f(x)| = g(x).$$

6. The height, in meters, of a rocket above the ground t seconds after launch was given by the function

   $$h(t) = 400t - 9.8t^2$$

(a) Determine the time t at which the rocket's instantaneous velocity was 100 meters per second.

$t = 15.306122$ seconds

(b) Find an interval of time starting at $t = 5$ during which the average velocity of the rocket was 100 meters per second. Give the length of this interval.

The interval should have length 20.6122 seconds.

7. Determine the value of a and the value of b such that the following function, f, is continuous at $x = 0$.

$$f(x) = \begin{cases} \frac{\sqrt{ax+b}-5}{x} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Since

$$\lim_{x\to 0^+} f(x) = 1,$$

we need

$$\lim_{x\to 0^-} f(x) = 1$$

in order for f to be continuous at 0. Since $\lim_{x\to 0^-} x = 0$, if

$$\lim_{x\to 0^-} \frac{\sqrt{ax+b}-5}{x}$$

exists, then it must be the case that

$$\lim_{x\to 0^-} \left(\sqrt{ax+b}-5\right) = 0.$$

But,

$$\lim_{x\to 0^-} \left(\sqrt{ax+b}-5\right) = \sqrt{b}-5$$

and so we may conclude that $b = 25$.

Then we have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sqrt{ax+25}-5}{x} = \lim_{x\to 0^-} \frac{a}{x\left(\sqrt{ax+25}+5\right)} = \lim_{x\to 0^-} \frac{a}{\sqrt{25}+5} = \frac{a}{10} = 1$$

and so $a = 10$.

Thus, if $a = 10$ and $b = 25$, then f will be continuous at $x = 0$, since then

$$f(0) = \lim_{x\to 0} f(x) = 1.$$