Math 124I - Winter 2003
Mid-Term Exam Number Two - Solutions
February 20, 2003

1. Find $\frac{dy}{dx}$. You need not simplify your result.

   (a) $y = \left(x^3 - 2x + \cos x\right)^8$

   Solution: $\frac{dy}{dx} = 8\left(x^3 - 2x + \cos x\right)^7 \left(3x^2 - 2 - \sin x\right)$

   (b) $y = \frac{x^3 + 4}{x^2 - x + 1}$

   Solution: $\frac{dy}{dx} = \frac{(3x^2)(x^2 - x + 1) - (x^3 + 4)(2x - 1)}{(x^2 - x + 1)^2}$

   (c) $y = \sec(x + e^x)$

   Solution: $\frac{dy}{dx} = (\sec(x + e^x)\tan(x + e^x))(1 + e^x)$

   (d) $y = x \sin 2x$

   Solution: $\frac{dy}{dx} = (1)\sin 2x + x(\cos 2x)(2) = \sin 2x + 2x \cos 2x$

2. Find $\frac{dy}{dx}$. You need not simplify your result.

   (a) $y = \ln \ln x$

   Solution: $\frac{dy}{dx} = \frac{1}{\ln x}\frac{1}{x} = \frac{1}{x \ln x}$

   (b) $x + \sin y = y + \cos x$.

   Solution: $1 + (\cos y)\, y' = y' - \sin x$ so $y'(\cos y - 1) = -1 - \sin x$ and hence $\frac{dy}{dx} = y' = \frac{-1 - \sin x}{\cos y - 1}$.

3. Suppose $f(x) = (3 - 5x)^{-2}$. Find $f'''(0)$.

   Solution:

$$f'(x) = (-2)(3 - 5x)^{-3}(-5) = (-2)(-5)(3 - 5x)^{-3}$$

$$f''(x) = (-2)(-5)(-3)(3 - 5x)^{-4}(-5) = (-2)(-5)(-3)(-5)(3 - 5x)^{-4}$$

$$f'''(x) = (-2)(-5)(-3)(-5)(-4)(3 - 5x)^{-5}(-5)$$

$$= (-2)(-5)(-3)(-5)(-4)(-5)(3 - 5x)^{-5} = 3000(3 - 5x)^{-5}$$

so

$$f'''(0) = 3000(3)^{-5} = \frac{3000}{243} = \frac{1000}{81}.$$

4. Find an equation of the tangent line to the curve

$$y = \frac{\cos x}{1 + e^x}$$

at the point $\left(0, \frac{1}{2}\right)$.

Solution:

$$y' = \frac{(-\sin x)(1 + e^x) - (\cos x)(e^x)}{(1 + e^x)^2}$$

so at $x = 0$, $y' = \frac{0 - 1}{2^2} = -\frac{1}{4}$. Hence the equation of the tangent line is

$$y - \frac{1}{2} = -\frac{1}{4}(x - 0) = -\frac{1}{4}x, \text{ i.e., } y = -\frac{1}{4}x + \frac{1}{2}.$$

5. Suppose $g(x) = \frac{xf(x)}{1 + h(x)}$. Find $g'(2)$ given that:

$$f(2) = 1,\ f'(2) = 0,\ h(2) = -2, \text{ and } h'(2) = 3.$$

Solution:

$$g'(x) = \frac{((1)f(x) + xf'(x))\,(1 + h(x)) - (xf(x))h'(x)}{(1 + h(x))^2}$$

so

$$g'(2) = \frac{(f(2) + 2f'(2))(1 + h(2)) - 2f(2)h'(2)}{(1 + h(2))^2}$$

$$= \frac{(1 + 2 \cdot 0)(1 + (-2)) - 2(1)(3)}{(1 + (-2))^2} = -7.$$

6. Find a parabola with equation $y = ax^2 + bx$ whose tangent line at $(2, 14)$ is $y = 17x - 20$.

Solution: If $(2, 14)$ is on the curve then,

$$14 = a(2)^2 + b(2) = 4a + 2b$$

i.e, $2a + b = 7$. Also, since the line $y = 17x - 20$ is tangent to the curve at $x = 2$,

$$17 = y'(2).$$

Since $y' = 2ax + b$, $y'(2) = 4a + b$, so

$$4a + b = 17.$$

Combining this with $7 = 2a + b$, you get

$$2a = 10, \text{ so } a = 5.$$

Hence $b = 7 - 2a = -3$, and the parabola is $y = 5x^2 - 3x$.

7. Find the equations of the tangent lines to $y = (\ln x)^2$ which passes through the origin.

Solution: Suppose the tangent line is tangent to the curve at $x = a$. Then the slope of the line is

$$\frac{(\ln a)^2}{a}$$

since it passes through the origin. Also, $y' = \dfrac{2\ln x}{x}$, so the slope is also

$$\frac{2\ln a}{a}.$$

Equating these two, we have

$$\frac{(\ln a)^2}{a} = \frac{2\ln a}{a}$$

so

$$\ln a = 2 \text{ or } \ln a = 0$$

and hence

$$a = e^2 \text{ or } a = 1.$$

The slope of the first line is $\dfrac{(\ln e^2)^2}{e^2} = \dfrac{4}{e^2}$, so the line has equation

$$y - (\ln e^2)^2 = \frac{4}{e^2}(x - e^2)$$

which is

$$y = \frac{4}{e^2}(x - e^2) + 4.$$

The other line has slope zero, so its equation is

$$y = 0.$$